A MODEL INVESTIGATING INCENTIVES FOR ILLEGAL CULTURAL GOODS SHARING SITES TO BECOME LEGAL

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Abstract

We build on parts of the elegant model proposed in [1] to investigate the possibility of legalizing illegal sites sharing cultural goods in exchange of a retribution.

1 Context

Digital cultural goods, including music, movies and books, are subject to intense illegal circulation over the Internet. Various technologies are used for this purpose, e.g. DDL, Streaming or BitTorrent. Legal responses to reduce such practices have shown limited efficiency, in spite of a few largely publicized cases. The aim of this work is to investigate another path to let producers of cultural goods recover some of the retribution they are entitled to. The idea is to provide incentive for (some) illegal forums to become legal in exchange of a monetary compensation, proportional to the amount of downloaded goods. In order to study the economic feasibility of this proposition[1], we set up a simple model for the circulation of cultural goods and the financial fluxes it produces. Under some assumptions on the behaviour of consumers, we compute the profits of different actors such that the industry, the artists or illegal forums, both in the current situation and in a framework where a proportional voluntary retribution would be paid by forums willing to become legal. This allows us to highlight scenarios where both the industry and forums would financially benefit from such a shift.

The remaining of this note is organized as follows. In Section 2, we present our model in an abstract setting: we formulate some general assumptions on the structure of the different actors (industry, consumers, forums, . . .) in Section 2.1, which enable us to compute in Section 2.2 the profits, depending on various relevant parameters such as for instance the price of the goods or the tendency for a given consumer to resort to pirate downloading. This analysis provides general bounds on the retribution that illegal forums would accept to pay in exchange of legalisation. These bounds characterize the situations where each actor sees its profits increase. Then, in Section 3, we perform explicit computations in three cases of interest describing more precisely the structure of both the set of consumers and the one of cultural goods. Numerical calculations allow us to provide acceptable ranges of values for the retribution. Finally, Section 4 proposes some extensions to our simple model that would be desirable to make it more complete.

[1] we do not consider in this work technical or legal aspects.

2 An abstract model

2.1 General assumptions

Our model is an abstract and generalized version of a part of the one presented in [1], which deals with the specific case of the music industry.

We consider cultural goods of a given type, e.g. music, movies, video games or books. Four to six kind of actors are involved:

1. the industry, which produces and sells the cultural goods;
2. consumers;
3. legal digital platforms, that propose physical or digital versions of the goods for purchase. In the case of the musical or book industry, this corresponds for instance to Amazon.com or Fnac.com;
4. illegal forums for downloading digital versions of the goods.

In some instances (e.g. music), one or two more types of actors are present:

5. platforms providing free and legal streaming services and paying subscriptions. Typical examples include Deezer or Spotify;
6. the artists.

In this preliminary study, only actors 1,2, and 4 are taken into account. In addition, for the sake of simplicity, we consider a monopolistic situation for actor 1, so that we will speak of "the" industry I.

Let us now detail the structure of the set of actors.

<u>Industry and cultural goods</u>

We consider the situation at a given time T. The set G of cultural goods produced by industry at this particular time is a countable set of points distributed on $2\,\mathbb{N}$, the set of even integers[2]. Each cultural good is identified with its coordinate on the positive real line. These coordinates are denoted $g_i = 2i, i = 0, 1, \ldots$. Each good is assumed to be sold at the same price p.

The global cost incurred by industry to produce and promote all cultural goods is denoted K^I. Although we will not need to detail it, we may assume, similarly to [1], that it comes from three sources: fixed costs, denoted K^I_F, production costs that depend on the structure of G and are denoted K^I_P, and promotional costs. These latter costs depends on the effort x_i made to promote good g_i. An effort x_i translates into a cost x_i^β with $\beta > 1$. As a consequence,

$$K^I = b \sum_i x_i^\beta + K^I_G + K^I_F, \tag{1}$$

(i will always be used for indexing goods, so that a symbol like $\sum_i$ means that the sum runs over all goods g_i in G).

[2] This choice is purely for notational convenience; it helps avoiding unsightly formulas later.

Each good g_i is characterized by its “attractiveness” a_i, a real number in $[0, 1]$. The exact meaning of attractiveness and the choice of its range will become clear below. In [1], attractiveness is chosen to be of the form aq_i, where the subjective “quality” q_i perceived by consumers depends on the effort x_i in the following way:

$$q_i = x_i^\alpha, \tag{2}$$

with $\alpha \in (0, 1)$ (since quality increases less and less when promotional effort increases). Again, we will have no need of this expression below.

Consumers

Consumers are distributed on the infinite strip $[0, 1] \times \mathbb{R}$. Each consumer C is characterized by its coordinates (θ, x), which are interpreted as follows:

- the distance $\xi = \xi(x) := d(C, G) = \inf_i |x - g_i|$ measures the willingness of consumer $C(x, \theta)$ to consume a good: the smallest $d(C, G)$, the more the consumer is interested in the production. Note that $\xi \in [0, 1]$. In addition, each consumer buys or pirates only the good that is “closest” to him at time T (or chooses the one with smallest coordinate if there are two such goods);
- the coordinate θ characterizes the behaviour of C as regards pirating. We assume that all goods are available for free on illegal forums. In addition, following [1], we consider that pirating a good translates into an immaterial cost for the consumer. This immaterial cost consists of two components: a "comfort cost’, that incorporates at least the following elements: (a) a downloaded good will typically be of lower technical quality, (b) it may not be as easily accessible as the original version, (c) cover and other information may be unavailable, (d) downloading from a forum generally requires more technical skills. The second component is a "legal cost", that accounts for the fact that typical consumers prefer to remain legal rather than facing the possibility of being prosecuted. Both costs are combined and their effect is taken into account as follows: we assume that the immaterial cost may be written as $\frac{\theta}{\tau}\, p$, where θ follows a distribution whose support is $[0, 1]$ and τ is a real in $(0, 1)$. The parameter θ measures how reluctant a particular consumer is with regard to piracy: when $\theta < \tau$, the consumer is willing to use illegal downloading, since its immaterial cost is smaller than industry cost p, while values of θ larger than τ prevent piracy.

The behaviour of consumer C is described by the maximal amount $w = w(x)$ that he is prepared to pay to buy good g_i, where $i = i(x)$ is the element of G that is closest to him. This amount depends on attractiveness a_i and on ξ as follows (we write i instead of $i(x)$ for simplicity):

$$w = a_i - \xi. \tag{3}$$

Each consumer is faced with three choices:

- if $p < \frac{\theta}{\tau}p$ and $p < w$, then he will buy the good;
- if $\frac{\theta}{\tau}p = \min(p, \frac{\theta}{\tau}p, w)$, then he will illegally download the good;
- if $\min(p, \frac{\theta}{\tau}p, w) = w$, then he will pass.

These situations are equivalently characterized by the following conditions:

1. buy if $\tau < \theta \leqslant 1$ and $\xi < a_i - p$,

2. pirate if $0 \leqslant \theta < \tau$ and $\xi < a_i - \frac{\theta}{\tau}p$,

3. no consumption if $a_i - \min(1, \frac{\theta}{\tau})p < \xi$.

Note that case 1 is void whenever $a_i < p$; likewise, case 2 is void if $a_i < \frac{\theta}{\tau}p$.

As in [1], we assume that, when there is consumption, the consumer devotes a share $0 \leqslant \rho \leqslant 1$ of what he has left in consuming products related to the good he has bought or pirated[3]: this amounts to $\rho(w-p)$ in the case he has bought the good and to ρw if he used an illegal downloading site. From these amounts, industry collects a fraction σ (where $0 \leqslant \sigma \leqslant \rho$), that is $\sigma(w-p)$ when the good was bought and σw when it was pirated.

The distribution of consumers on $[0,1] \times \mathbb{R}$ is described by a measure $\mu = \mu(d\theta, dx)$ on this set. We will mainly consider three configurations that seem to be of particular interest:

1. Configuration 1 is when $\mu(d\theta, dx) = \mu_\theta(d\theta)\mu_x(dx)$ with μ_θ uniform on $[0,1]$ and μ_x uniform on $[-1, 2n+1]$ for some fixed positive integer n. More precisely, we set $\mu_\theta(d\theta) = d\theta \mathbf{1}_{\{\theta\in[0,1]\}}$ and $\mu_x(dx) = c\,\frac{dx}{2n+2}\mathbf{1}_{\{x\in[-1,2n+1]\}}$, where $c > 0$ is a constant allowing to tune the total number of consumers. In this model, consumers are thus only interested in the first n goods, and the proportion of pirates is equal to τ. If one assumes in addition, as we will, that $a_i = a$ for all i, then nothing depends on i: all goods are equally popular, and the proportion of consumers that will buy or pirate g_i is the same for all i. This is essentially the case considered in [1].

2. Configuration 2 takes again $\mu(d\theta, dx) = \mu_\theta(d\theta)\mu_x(dx)$ with μ_θ uniform on $[0,1]$, but with μ_x supported on $\mathbb{R}^+$ and given by $\mu_x(dx) = c\,\frac{dx}{1+x^\gamma}$, $c > 0, \gamma > 1$. This form serves as rough model for the situation where there are a few popular goods with rapidly decreasing popularity, and "infinitely" many goods with very low popularity. Such Pareto type distributions are often used for this purpose.

3. Configuration 3 allows for a coupling between willingness to pirate and popularity of a good. Assuming that more popular goods are more prone to pirating, we wish to design μ so that, for values of x such that $\mu(d\theta, dx)$ is large (*i.e.* the desired good is popular), then there is more mass when θ close to 0 (*i.e.*, the proportion of pirates increases). Writing $\mu(d\theta, dx) = \mu_x(dx)\,\mu_\theta(d\theta|x)$, a simple choice is to set $\mu_\theta(d\theta|x) = \left((1-\theta)\frac{\mu_x(dx)}{dx} + 1 - \frac{\mu_x(dx)}{2dx}\right) d\theta$, that is, the $\theta-$marginal density is a line with decreasing slope equal to $-\frac{\mu_x(dx)}{dx}$ such that $\int_0^1 \mu_\theta(d\theta|x) = 1$ for all x. Simple computations show that, if $\mu_x(dx)$ is chosen as in Configuration 2, then the proportion of pirates among consumers with first coordinate between x and $x + dx$, that is $\frac{\int_0^\tau \mu_\theta(d\theta|x)}{\int_0^1 \mu_\theta(d\theta|x)}dx$, is equal to $\frac{\tau(2x^\gamma+3-\tau)}{2(1+x^\gamma)}dx$. For the most popular good, *i.e.* when $x = 0$, this is $\frac{\tau(3-\tau)}{2}$, which is indeed larger than τ, the proportion of pirates for the least popular goods (which corresponds to $x \to \infty$). In general, our choice for $\mu_\theta(d\theta|x)$ ensures that, if the support of μ_x extends to infinity, then the limiting proportion of pirates when x tends to infinity is exactly τ. Other, non linear, choices are of course possible.

[3]these can be live music, derived products for movies or video games, ...

Forums

We assume that there exist m forums $F_1, \ldots, F_m$, and that consumers that choose to use illegal download are drawn at random to one of the F_k with probability z_k. Forum F_k (the index k will be reserved for forums) is characterized by a parameter λ_k measuring its reluctance to become legal. Indeed, some forums might agree to become legal under certain conditions, while others would never do so (in that case, λ is infinite). Without loss of generality, we assume $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m$. The larger λ, the more reluctant is the forum to become legal.

We wish to model the revenue of illegal forums. A first very rough way to do so is as follows. Forums earn money from advertisements. We assume that they get paid a fixed amount $\tilde{p}$ for each download (although a more correct model would assume that they get paid for each unique visitor to their page). The total profit P_{IF} is then obtained by integrating over all pirate downloads.

Legalizing illegal forums could be performed by asking them to pay the industry a share of their revenue. We choose a form $r_i = r_i(g_i) > 0$ for each download of good g_i for this contribution. Since we assume that all goods are equally available on all forums, the contribution will essentially be equal $r = \sum_i pop(i) r_i$, where $pop(i)$ is the normalized popularity of g_i (normalized means that the sum of all $pop(i)$ is one), Under the assumption of a uniform distribution of x, $pop(i)$ does not depend on i.

Forums that will agree to become legal are then those for which λ_k is smaller that a threshold $\lambda(r)$, a function of the required contribution r. We let L denote the number of such forums, so that $\lambda_L < \lambda(r) \leqslant \lambda_{L+1}$. A forum that becomes legal will typically attract more advertisements, and, more importantly, from legal companies. This will likely increase the amount it gets paid for each download (visitor). We denote this new amount by $\hat{p}$, with $\hat{p} > \tilde{p}$.

For legal forums, there is no more a notion of pirating a good. Legalizing some forums would then seem to imply that this form of consumption will kill both buying cultural goods and pirating them. Indeed, consumers will have access for free to legal goods. This is however not the case. On the one hand, a consumer downloading from a legalized forum is still faced with the comfort cost part of the immaterial cost. This cost will make some consumers still reluctant to use forums, even though they are legalized, and prefer to buy the good. We choose to model the comfort cost in a manner similar to the one of the total immaterial cost of pirating, i.e. in the form $\frac{\theta}{\hat{\tau}} p$. We choose $\hat{\tau}$ in $(0, 1)$ with $\tau \leqslant \hat{\tau} \leqslant 1$ since the immaterial cost here is less than for pirating (the legal cost has been removed). On the other hand, pirating will still occur, at least in a first phase for the following reason: if a consumer has a choice of downloading the same good with the same features (quality, ...) either on a legalized forum or on a pirate one, he will obviously choose the legalized forum (except in rare occasions that we choose to ignore). This simply translates the fact that the cost $\frac{\theta}{\hat{\tau}} p$ is always smaller than $\frac{\theta}{\tau} p$ since $\hat{\tau} \geqslant \tau$. However, as was assumed above, when searching for a cultural good, the consumer is redirected at random to one of the m forums. In our model, a consumer thus never has a choice between a legalized and a pirate forum: a proportion $\sum_{k=1}^{L} z_k$ of consumers will be redirected to a legalized forum, and thus will have a choice between (a) this kind of consumption, (b) buying, and (c) no consumption, while a proportion $1 - \sum_{k=1}^{L} z_k$ of consumers will have a choice between (a) pirating, (b) buying, and (c) no consumption. It might be argued that, in time, the legalized offer will be better known and will attract more consumers. This can be accounted for in our model by adjusting dynamically the probabilities $(z_k)_k$. We will typically assume that, prior to legalization, all forums have equal probability of being chosen, *i.e.* $z_k = 1/m$ for all k, and that this remains the case for some time after legalization of the first L forums. This assumption is justified by the fact that it will take some time before consumers get to be

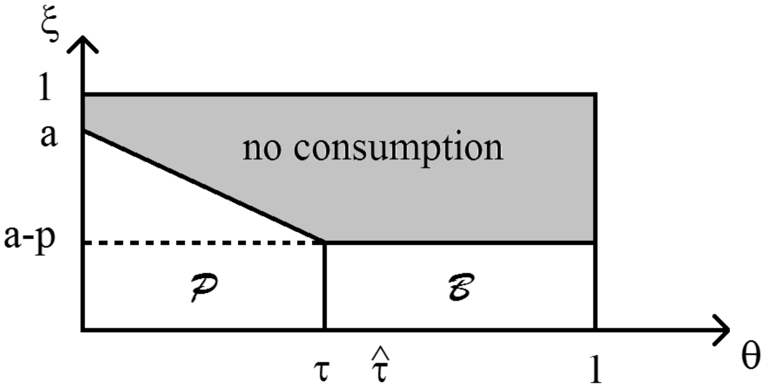

Figure 1: Sets of buyers $\mathcal{B}$, pirates $\mathcal{P}$, and non-consumers for one good with attractivity a.

acquainted to this new form of downloading and to adjust their practice. An extension to this work will examine the case where $z_1, \dots, z_L$ become larger than $z_{L+1}, \dots, z_m$.

After legalization, we thus have three non void classes of consumers: pirates, downloaders from legalized forums, and buyers. We assume that the distribution of consumers does not change after legalization: indeed, their preference as regards the goods, as encoded in the x variable, has no reason to be altered, while their changed behaviour with respect to pirating is already taken into account through the introduction of $\hat{\tau}$.

We now compute the profits of the different actors, prior to, and after legalization.

2.2 Computing profits

We shall make use of the following notations. $\mathcal{B}$ denotes the set of buyers, $\mathcal{P}$ the set of pirates and $\mathcal{C} = \mathcal{B} \cup \mathcal{P}$ the set of active consumers before legalization. After legalization of some forums, the set of active consumers $\overline{\mathcal{C}}$ is split as $\overline{\mathcal{C}} = \overline{\mathcal{B}} \cup \mathcal{D} \cup \overline{\mathcal{P}}$ where $\mathcal{D}$ is the subset of downloaders from legalized forums, $\overline{\mathcal{B}}$ and $\overline{\mathcal{P}}$ are respectively the new sets of buyers and pirates after legalization. The sets $\mathcal{C}, \overline{\mathcal{C}}$ and some of their subsets are pictured on Figures 1 to 3.

Let us express various measures and moments that will be needed in the sequel. First,

$$\mu(\mathcal{B}) = \sum_i \int_\tau^1 \int_{2i-a_i+p}^{2i+a_i-p} \mu(d\theta, dx), \tag{4}$$

with the convention that any integral whose lower bound is not smaller than its upper bound is considered to be 0 (remark also that the integrals with respect to x above are non-overlapping since $a_i + a_{i+1} \leqslant 2 \leqslant 2p + 2$). Likewise,

$$\mu(\overline{\mathcal{B}}) = \left(1 - \frac{L}{m}\right) \mu(\mathcal{B}) + \frac{L}{m} \sum_i \int_{\hat{\tau}}^1 \int_{2i-a_i+p}^{2i+a_i-p} \mu(d\theta, dx), \tag{5}$$

$$\mu(\mathcal{P}) = \sum_i \int_0^\tau \int_{2i-a_i+\frac{\theta}{\tau}p}^{2i+a_i-\frac{\theta}{\tau}p} \mu(d\theta, dx), \tag{6}$$

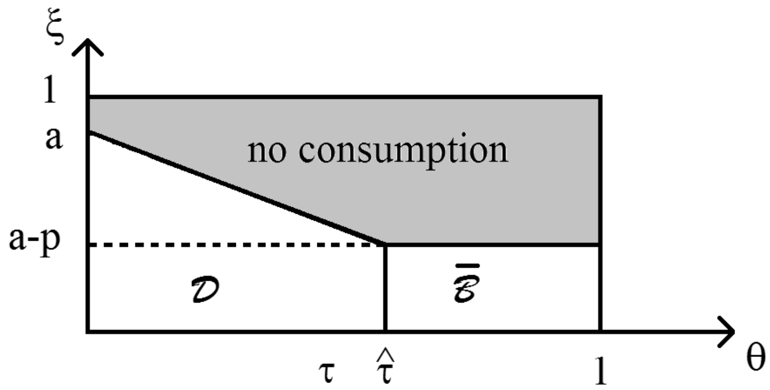

Figure 2: Sets of buyers $\overline{\mathcal{B}}$, downloaders from a legalized forum $\mathcal{D}$, and non-consumers for one good with attractivity a for individuals that were redirected to a legalized forum.

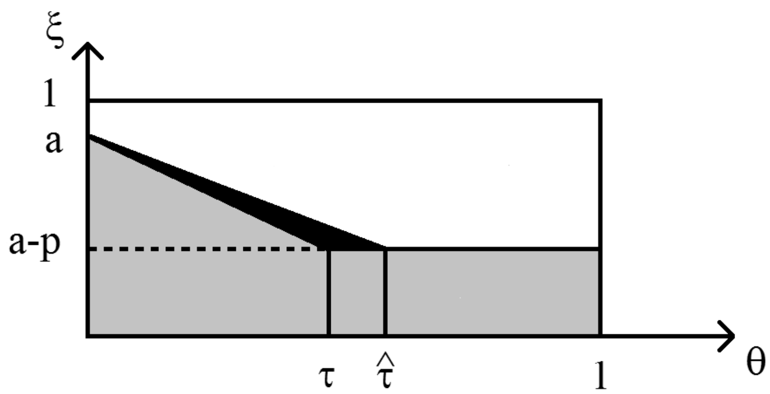

Figure 3: Sets $\mathcal{C}$ (gray) and $\overline{\mathcal{C}}$ in the case of a legalized forum (gray + black) for a particular good with attractivity a.

$$\mu(\overline{\mathcal{P}}) = \left(1 - \frac{L}{m}\right) \mu(\mathcal{P}), \tag{7}$$

$$\mu(\mathcal{D}) = \frac{L}{m} \sum_i \int_0^{\hat{\tau}} \int_{2i-a_i+\frac{\theta}{\hat{\tau}}p}^{2i+a_i-\frac{\theta}{\hat{\tau}}p} \mu(d\theta, dx), \tag{8}$$

$$\int_{\mathcal{B}} w\, \mu(d\theta, dx) = \sum_i \int_\tau^1 \int_{2i-a_i+p}^{2i+a_i-p} (a_i - |x - 2i|)\, \mu(d\theta, dx), \tag{9}$$

$$\int_{\overline{\mathcal{B}}} w\, \mu(d\theta, dx) = \left(1 - \frac{L}{m}\right) \int_{\mathcal{B}} w\, \mu(d\theta, dx) + \frac{L}{m} \sum_i \int_{\hat{\tau}}^1 \int_{2i-a_i+p}^{2i+a_i-p} (a_i - |x - 2i|)\, \mu(d\theta, dx), \tag{10}$$

$$\int_{\mathcal{P}} w\, \mu(d\theta, dx) = \sum_i \int_0^\tau \int_{2i-a_i+\frac{\theta}{\tau}p}^{2i+a_i-\frac{\theta}{\tau}p} (a_i - |x - 2i|)\, \mu(d\theta, dx), \tag{11}$$

$$\int_{\overline{\mathcal{P}}} w\, \mu(d\theta, dx) = \left(1 - \frac{L}{m}\right) \int_{\mathcal{P}} w\, \mu(d\theta, dx) \tag{12}$$

$$\int_{\mathcal{D}} w\, \mu(d\theta, dx) = \frac{L}{m} \sum_i \int_0^{\hat{\tau}} \int_{2i-a_i+\frac{\theta}{\hat{\tau}}p}^{2i+a_i-\frac{\theta}{\hat{\tau}}p} (a_i - |x - 2i|)\, \mu(d\theta, dx). \tag{13}$$

The profit P_I of industry prior to legalizing forums may be expressed as follows:

$$\begin{aligned} P_I &= \int p_u(\theta, \xi)\, \mu(d\theta, dx) - K_I \\ &= \int_{\mathcal{B}} [p + \sigma(w - p)]\ \mu(d\theta, dx) + \int_{\mathcal{P}} \sigma w\, \mu(d\theta, dx) - K_I \\ &= (1 - \sigma) p \mu(\mathcal{B}) + \sigma \int_{\mathcal{C}} w\, \mu(d\theta, dx) - K_I \end{aligned} \tag{14}$$

where p_u is the profit per individual.

Post-legalization industry profit reads:

$$\begin{aligned} \overline{P_I} &= \int_{\overline{\mathcal{B}}} [p + \sigma(w - p)]\ \mu(d\theta, dx) + \int_{\mathcal{D} \cup \overline{\mathcal{P}}} \sigma w\, \mu(d\theta, dx) + r\ \mu(\mathcal{D}) - K_I \\ &= (1 - \sigma) p \mu(\overline{\mathcal{B}}) + \sigma \int_{\overline{\mathcal{C}}} w\, \mu(d\theta, dx) + r\ \mu(\mathcal{D}) - K_I. \end{aligned} \tag{15}$$

The incurred profit or loss induced by legalization for industry is

$$\overline{P_I} - P_I = (1 - \sigma) p \left(\mu(\overline{\mathcal{B}}) - \mu(\mathcal{B})\right) + \sigma \left(\int_{\overline{\mathcal{C}}} w\, \mu(d\theta, dx) - \int_{\mathcal{C}} w\, \mu(d\theta, dx)\right) + r\ \mu(\mathcal{D}). \tag{16}$$

Using (4) to (13), one computes:

$$\begin{aligned} \overline{P_I} - P_I \quad &= \quad -\frac{L}{m} \sum_i \int_\tau^{\hat{\tau}} \int_{2i-a_i+p}^{2i+a_i-p} [p(1 - \sigma) + \sigma(a_i - |x - 2i|)]\, \mu(d\theta, dx) \\ &+ \quad \sigma \frac{L}{m} \left(\sum_i \int_0^{\hat{\tau}} \int_{2i-a_i+\frac{\theta}{\hat{\tau}}p}^{2i+a_i-\frac{\theta}{\hat{\tau}}p} (a_i - |x - 2i|)\, \mu(d\theta, dx) - \sum_i \int_0^\tau \int_{2i-a_i+\frac{\theta}{\tau}p}^{2i+a_i-\frac{\theta}{\tau}p} (a_i - |x - 2i|)\, \mu(d\theta, dx) \right) \\ &+ \quad r\ \mu(\mathcal{D}). \end{aligned} \tag{17}$$

$\overline{P_I} - P_I$ is a sum of three terms. The one on the first line of (17) is non-positive since $\hat{\tau} \geqslant \tau$. The one on the second line is also typically non-positive. Thus, industry profit will increase only if r is large enough, which is intuitively obvious. Note that in the limiting case where $\hat{\tau} = \tau$, industry profit increases by the amount of $r\ \mu(\mathcal{D})$, as is expected.

Prior to legalization, profit of forum $k, k = 1, \ldots, m$, is equal to

$$P_{F_k} = \frac{1}{m} \int_{\mathcal{P}} \tilde{p}\, d\mu = \frac{\tilde{p}}{m} \mu(\mathcal{P}).$$

For $k = 1, \ldots, L$, profit after legalization becomes (recall that we assume equal probabilities $z_k, k = 1, \ldots, m$)

$$\overline{P_{F_k}} = \frac{1}{L} \int_{\mathcal{D}} \hat{p}\, d\mu - \frac{r}{L} \int_{\mathcal{D}} d\mu = \frac{\hat{p} - r}{L} \mu(\mathcal{D}),$$

while, for $k = L+1, \ldots, m$, the new profit reads

$$\overline{P_{F_k}} = \frac{1}{m-L} \int_{\overline{\mathcal{P}}} \tilde{p}\, d\mu = \frac{\tilde{p}}{m-L} \mu(\overline{\mathcal{P}}).$$

Thus, the shift in profit is

$$\overline{P_{F_k}} - P_{F_k} = \frac{\hat{p} - r}{L} \mu(\mathcal{D}) - \frac{\tilde{p}}{m} \mu(\mathcal{P}) \tag{18}$$

for $k = 1, \ldots, L$, and

$$\overline{P_{F_k}} - P_{F_k} = \tilde{p} \left(\frac{\mu(\overline{\mathcal{P}})}{m-L} - \frac{\mu(\mathcal{P})}{m} \right) = 0$$

for $k = L+1, \ldots, m$, where we have used (7). The fact that legalization has no impact on the profit of forums that choose to remain illegal is simply a consequence of our assumption that the z_k all remain equal after legalization.

Legalizing forums will be profitable to industry whenever (16) is positive, and to forums as soon as (18) is positive. This provides a range of values acceptable to both parties for the contribution r to be paid by forums to industry:

$$-\frac{1}{\mu(\mathcal{D})} \left((1-\sigma) p \left(\mu(\overline{\mathcal{B}}) - \mu(\mathcal{B}) \right) + \sigma \left(\int_{\overline{\mathcal{C}}} w\, \mu(d\theta, dx) - \int_{\mathcal{C}} w\, \mu(d\theta, dx) \right) \right) \leqslant r \leqslant \hat{p} - \tilde{p}\, \frac{L}{m}\, \frac{\mu(\mathcal{P})}{\mu(\mathcal{D})}. \tag{19}$$

Inequalities (19) constitute our master formula, from which all the analysis below derives.

At first sight, it seems that the upper bound on r in (19) is a decreasing function of L, which would be rather counter-intuitive. However, using (6) and (8), one sees that

$$\hat{p} - \tilde{p}\, \frac{L}{m}\, \frac{\mu(\mathcal{P})}{\mu(\mathcal{D})} = \hat{p} - \tilde{p}\, \frac{\sum_i \int_0^{\tau} \int_{2i - a_i + \frac{\theta}{\tau} p}^{2i + a_i - \frac{\theta}{\tau} p} \mu(d\theta, dx)}{\sum_i \int_0^{\hat{\tau}} \int_{2i - a_i + \frac{\theta}{\hat{\tau}} p}^{2i + a_i - \frac{\theta}{\hat{\tau}} p} \mu(d\theta, dx)},$$

which shows that it depends neither on L nor on m. Inspection of (8) and (17) reveals that the lower bound also does not depend on L nor on m.

Inequalities (19) are meaningful only if they define a non-empty interval in $\mathbb{R}^+$, *i.e.* if

$$\max\left(0,(1-\sigma)p\left(\mu(\mathcal{B})-\mu(\overline{\mathcal{B}})\right)+\sigma\left(\left(\int_{\mathcal{C}} w\,\mu(d\theta,dx)-\int_{\overline{\mathcal{C}}} w\,\mu(d\theta,dx)\right)\right)\right)<\hat{p}\,\mu(\mathcal{D})-\tilde{p}\,\frac{L}{m}\,\mu(\mathcal{P}). \tag{20}$$

One checks that, in the limiting case $\hat{\tau}=\tau$, $\frac{\mu(\mathcal{P})}{m}=\frac{\mu(\mathcal{D})}{L}$ so that (19) becomes:

$$0\leqslant r\leqslant \hat{p}-\tilde{p}.$$

In other words, would there be no legal costs, the upper bound on the contribution r would simply be the profit increase of legalized forums, a result which seems rather natural.

In the next section, we specialize the measure μ to the ones described in Configurations 1, 2, and 3 above and compute numerically the admissible range for r as given by (19), as a function of our various parameters.

3 Explicit computations

In all the numerical experiments below, we use the parameters displayed in Table 1 (not all parameters are relevant in each case).

c	n	σ	m	L	a	p	$\tilde{p}$	$\hat{p}$	τ	$\hat{\tau}$	γ
1	12	0.5	20	5	0.7	0.3	0.1	0.15	0.4	0.5	2

Table 1: Default values of the parameters used in numerical experiments.

3.1 Separable scenario

In the case where $\mu(d\theta,dx)=h(x)g(\theta)dxd\theta$, the various integrals computed above, and, as a consequence, (19), take a simpler form. Note first that we do not loose any generality in supposing then that $g\equiv 1$, since all we are interested in is the proportion of individuals with $\theta<\tau$. Replacing $\mu(d\theta,dx)$ by $h(x)dxd\theta$ and denoting H a primitive of h, one computes

$$\begin{aligned}
\mu(\mathcal{P}) &= \sum_i\int_0^\tau\int_{2i-a_i+\frac{\theta}{\tau}p}^{2i+a_i-\frac{\theta}{\tau}p} h(x)\,dx\,d\theta\\
&= \sum_i\int_0^\tau\left(H(2i+a_i-\frac{\theta}{\tau}p)-H(2i-a_i+\frac{\theta}{\tau}p)\right)\,d\theta\\
&= \tau\sum_i\int_0^1\left(H(2i+a_i-\theta p)-H(2i-a_i+\theta p)\right)\,d\theta\\
&= \tau IS(p),
\end{aligned}$$

where we have set $IS(p):=\sum_i\int_0^1\left(H(2i+a_i-\theta p)-H(2i-a_i+\theta p)\right)\,d\theta$. The same computations lead to

$$\mu(\mathcal{D})=\frac{L}{m}\,\hat{\tau}\,IS(p),$$

so that the upper bound in (19) now just reads $\hat{p} - \frac{\tau}{\hat{\tau}}\tilde{p}$. The lower bound does not simplify so drastically, but one can compute

$$
\begin{aligned}
\mu(\mathcal{B}) &= \sum_i \int_\tau^1 \int_{2i-a_i+p}^{2i+a_i-p} h(x)\, dx\, d\theta \\
&= (1-\tau)\sum_i (H(2i+a_i-p) - H(2i-a_i+p)) \\
&= (1-\tau)S(p),
\end{aligned}
$$

where $S(p) := \sum_i (H(2i+a_i-p) - H(2i-a_i+p))$. Likewise,

$$
\mu(\overline{\mathcal{B}}) = \left(\left(1-\frac{L}{m}\right)(1-\tau) + \frac{L}{m}(1-\hat{\tau})\right) S(p).
$$

Let $\check{H}_i$ denote a primitive of the function $x \mapsto (a_i - |x-2i|)h(x)$. Then

$$
\begin{aligned}
\int_{\mathcal{B}} w\mu(d\theta, dx) &= \sum_i \int_\tau^1 \int_{2i-a_i+p}^{2i+a_i-p} (a_i - |x-2i|)h(x)\, dx\, d\theta \\
&= (1-\tau)\sum_i \left(\check{H}_i(2i+a_i-p) - \check{H}_i(2i-a_i+p)\right) \\
&= (1-\tau)\check{S}(p),
\end{aligned}
$$

where $\check{S}(p) = \sum_i \left(\check{H}_i(2i+a_i-p) - \check{H}_i(2i-a_i+p)\right)$. One obtains in the same way

$$
\int_{\overline{\mathcal{B}}} w\mu(d\theta, dx) = \left(\left(1-\frac{L}{m}\right)(1-\tau) + \frac{L}{m}(1-\hat{\tau})\right) \check{S}(p),
$$

$$
\begin{aligned}
\int_{\mathcal{P}} w\mu(d\theta, dx) &= \sum_i \int_0^\tau \int_{2i-a_i+\frac{\theta}{\tau}p}^{2i+a_i-\frac{\theta}{\tau}p} (a_i - |x-2i|)h(x)\, dx\, d\theta \\
&= \tau\sum_i \int_0^1 \left(\check{H}_i(2i+a_i-\theta p) - \check{H}_i(2i-a_i+\theta p)\right)\, d\theta \\
&= \tau\widecheck{IS}(p),
\end{aligned}
$$

where we have set $\widecheck{IS}(p) := \sum_i \int_0^1 \left(\check{H}_i(2i+a_i-\theta p) - \check{H}_i(2i-a_i+\theta p)\right)\, d\theta$, and finally

$$
\int_{\mathcal{D}} w\mu(d\theta, dx) = \frac{L}{m}\hat{\tau}\widecheck{IS}(p).
$$

Gathering the above expressions, one sees that, in the separable case, (19) reads

$$
\left(1-\frac{\tau}{\hat{\tau}}\right)\frac{(1-\sigma)S(p) + \sigma(\check{S}(p) - \widecheck{IS}(p))}{IS(p)} \leqslant r \leqslant \hat{p} - \frac{\tau}{\hat{\tau}}\tilde{p}. \tag{21}
$$

These bounds hold for Configurations 1 and 2 that we investigate in the next two sections.

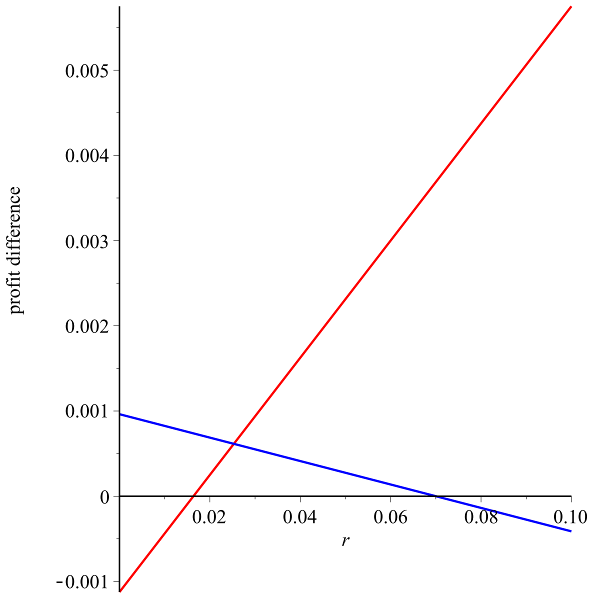

Figure 4: Profit difference for Industry (red) and legalized forums (blue) as a function of r.

3.2 Configuration 1

We first draw on Figure 4 the difference of profits before and after legalization for industry (Formula (16)) and legalized forums (Formula (18)) as a function of the retribution r.

Inspecting the bounds in (21), one sees that they do not depend on the parameter c. It is also easy to check that they do not depend on n since μ is uniform. We draw graphs displaying the behaviour of these bounds as functions of the remaining parameters, that is, $\sigma, a, p, \tilde{p}, \hat{p}, \tau, \hat{\tau}$, see Figures 5 to 11. The parameters that do not vary in each case are chosen as in Table 1. From Figure 5, one sees that the lower bound is tighter when σ is small, which is intuitively expected, since decreasing σ means that Industry will more severely suffer from pirating. Figure 6 indicates that the lower bound is a non-linearly increasing function of a, a fact which does not seem to possess a straightforward explanation. The evolution with respect to p, as displayed on Figure 7 is not easier to interpret. The behaviour of the upper bound as a function of $\tilde{p}$ and $\hat{p}$ is straightforward from (21) (Figures 8 and 9). Both the evolutions of the lower and upper bounds as functions of τ are also easy to interpret, see Figure 10. Note however that the slope of the upper bound is larger, in absolute value, than the one of the lower bound. Finally, the fact that both bounds are increasing non-linear functions of $\hat{\tau}$, as shown on Figure 11, is clear from (21). Notice that, of course, this time, the upper bound increases faster than the lower one.

3.3 Configuration 2

Here again, it is easy to check that the bounds in (21) do not depend on c. In addition, it is apparent that the upper bound coincides with the one in Configuration 1. For this reason, we do not draw the upper and lower bounds as in the previous section, but we rather display graphs comparing the evolutions of the lower bounds for Configurations 1 and 2 as functions of the relevant parameters, that is $\sigma, a, p, \tau, \hat{\tau}$.

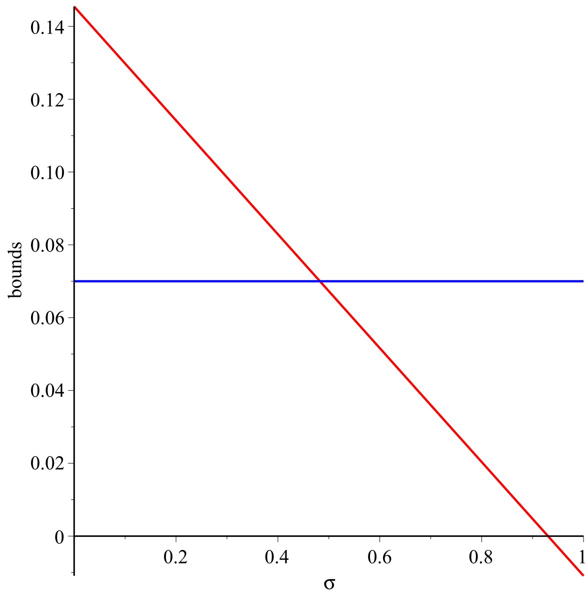

Figure 5: Lower bound (red) and upper bound (blue) in (21) as functions of σ.

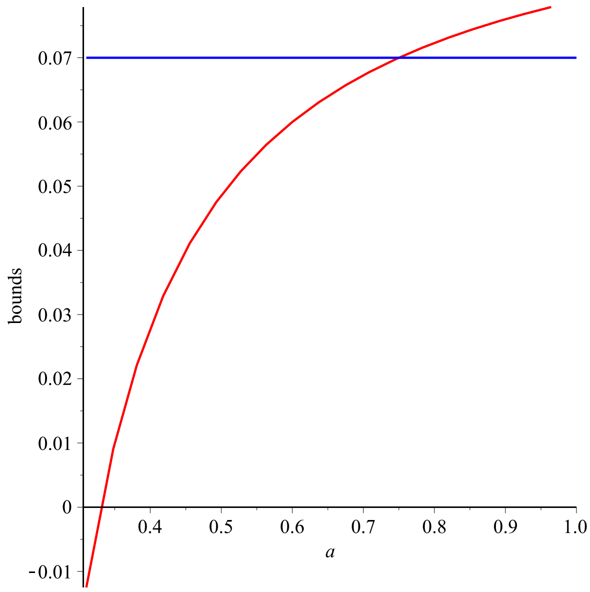

Figure 6: Lower bound (red) and upper bound (blue) in (21) as functions of a.

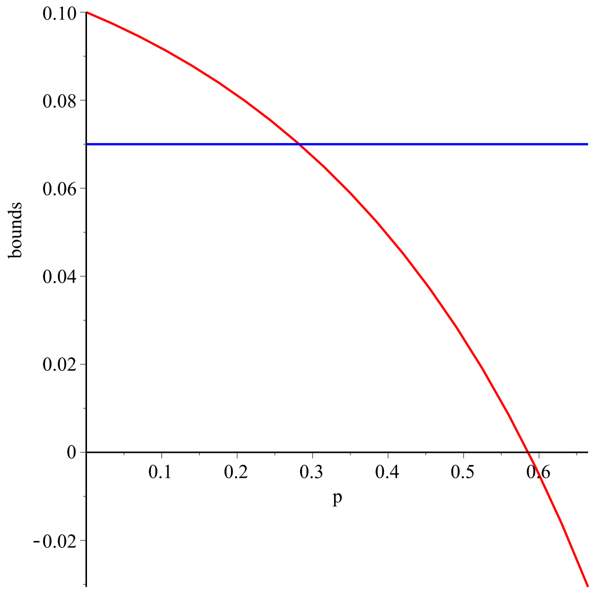

Figure 7: Lower bound (red) and upper bound (blue) in (21) as functions of p.

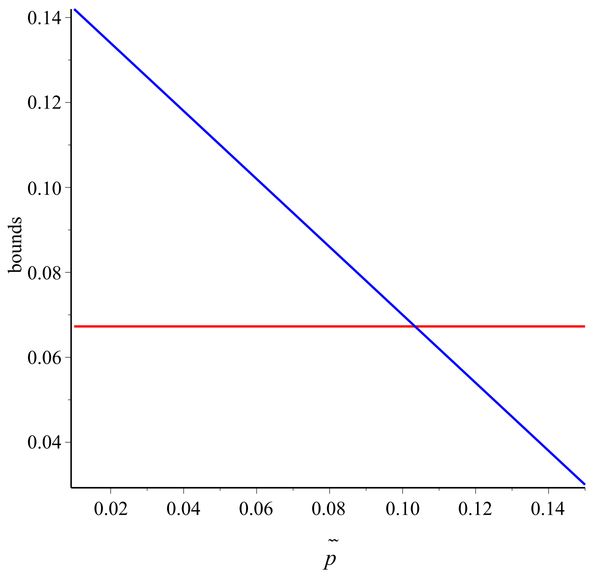

Figure 8: Lower bound (red) and upper bound (blue) in (21) as functions of $\tilde{p}$.

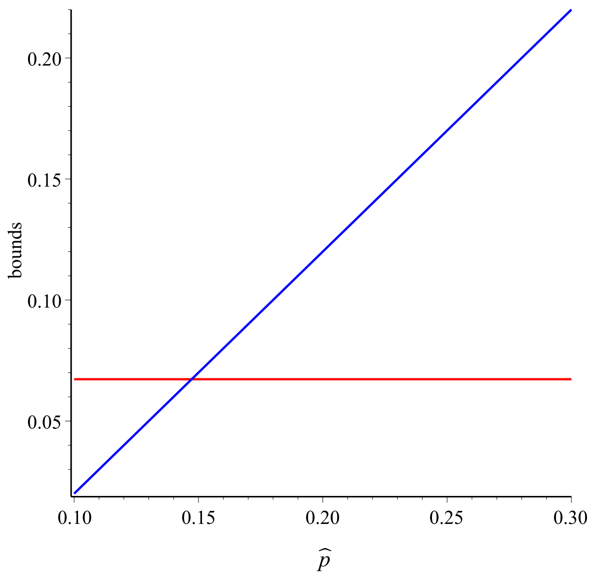

Figure 9: Lower bound (red) and upper bound (blue) in (21) as functions of $\hat{p}$.

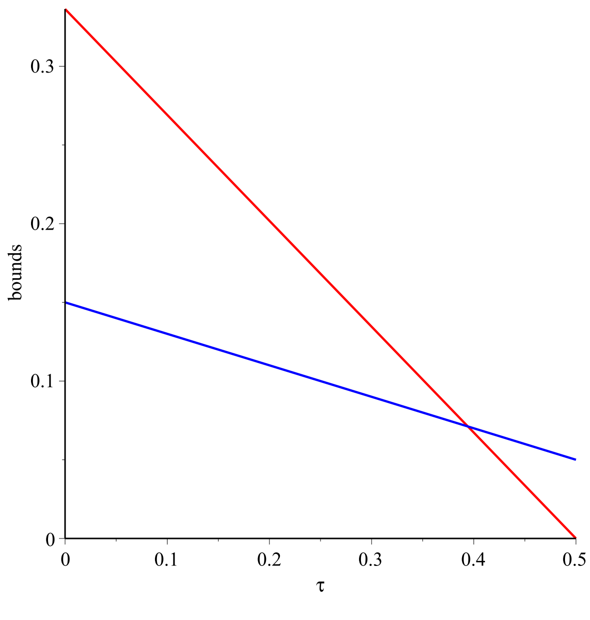

Figure 10: Lower bound (red) and upper bound (blue) in (21) as functions of τ.

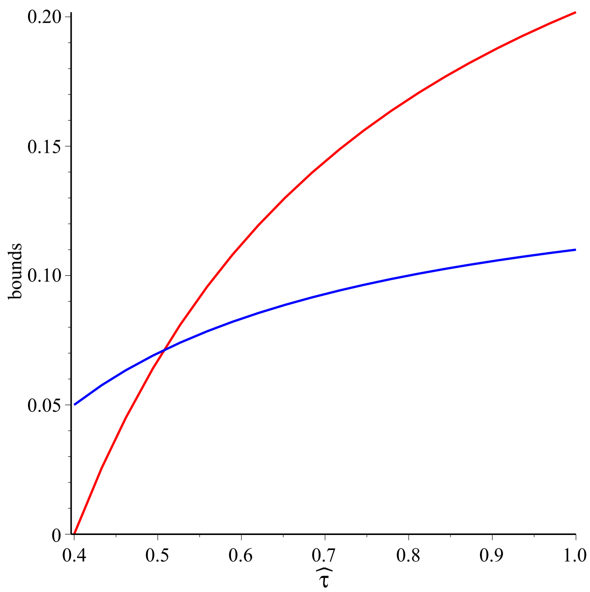

Figure 11: Lower bound (red) and upper bound (blue) in (21) as functions of $\hat{\tau}$.

These are on Figures 12 to 16. Two facts are noteworthy: first, for all graphs, the bounds are tighter in Configuration 2 than they were in Configuration 1 (the lower bound is larger): the (probably more realistic) modelling where one takes into account the fact that popularity varies wildly among goods entails stronger constraints on the retribution r; second, this increase of the lower bound is rather limited, indicating that our model is robust with respect to a change of distribution of consumers, at least as long as one remains in a separable case.

3.4 Configuration 3

Assuming the same form for the distribution in x as in Configuration 2, we set in this section

$$\mu(d\theta, dx) = \left(1 + \frac{c}{1 + x^\gamma}\left(\frac{1}{2} - \theta\right)\right) d\theta dx.$$

We draw on Figure 17 an example of the ruled surface which is the graph of the density of the measure. Note that, in this configuration, the bounds in (19) do depend on c.

We display the evolutions of the lower and upper bounds as functions of the various parameters (Figures 18 to 27). As one can see it from the graphs, the situation is here much more constrained than in the separable scenarios considered above. In particular, with the default values for the parameters, there are no acceptable values for the retribution r. Furthermore, in rather wide ranges of values, the lower bound remains larger than the upper one. Further studies are needed in order to determine which intervals for the parameters allow to determine feasible retributions, and whether these intervals correspond to realistic values.

of a.

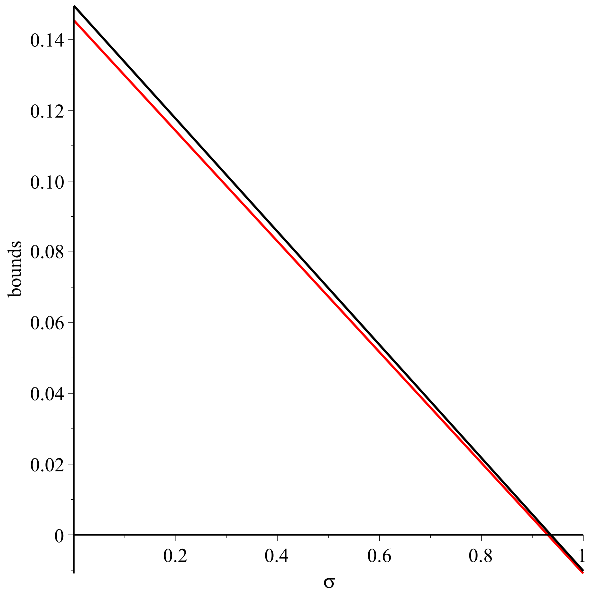

Figure 12: Comparison of the lower bound in Configurations 1 (red) and 2 (black) in (21) as functions of σ.

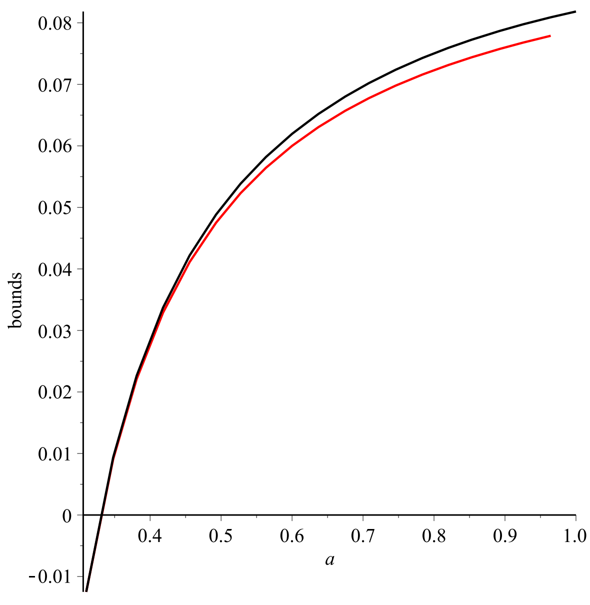

Figure 13: Comparison of the lower bound in Configurations 1 (red) and 2 (black) in (21) as functions of a.

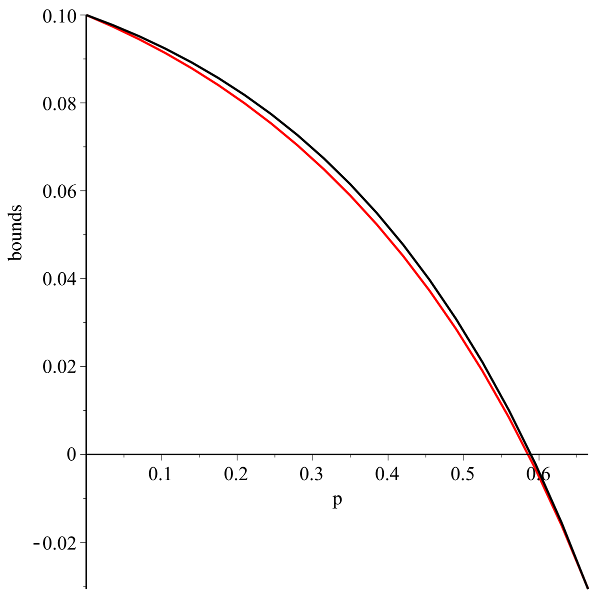

Figure 14: Comparison of the lower bound in Configurations 1 (red) and 2 (black) in (21) as functions of p.

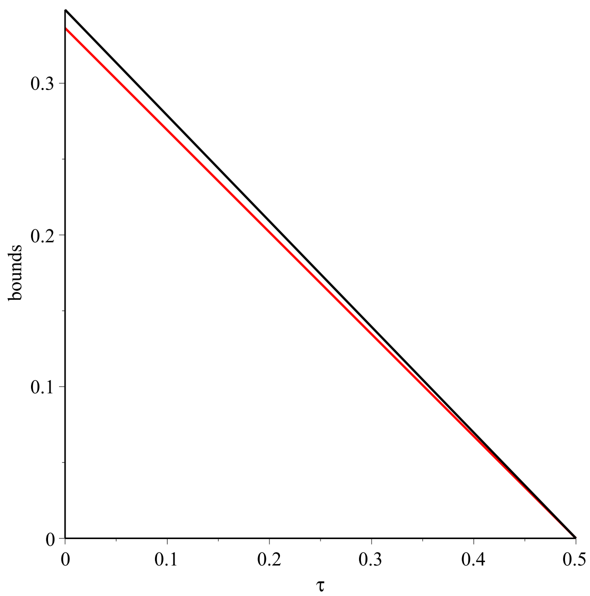

Figure 15: Comparison of the lower bound in Configurations 1 (red) and 2 (black) in (21) as functions of τ.

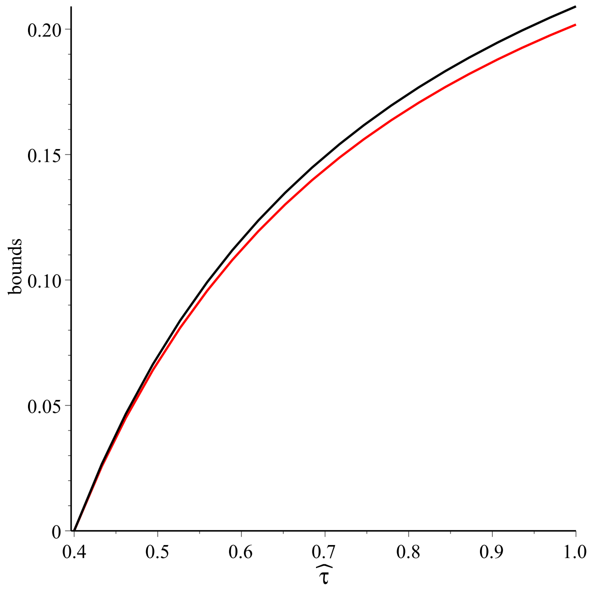

Figure 16: Comparison of the lower bound in Configurations 1 (red) and 2 (black) in (21) as functions of $\hat{\tau}$.

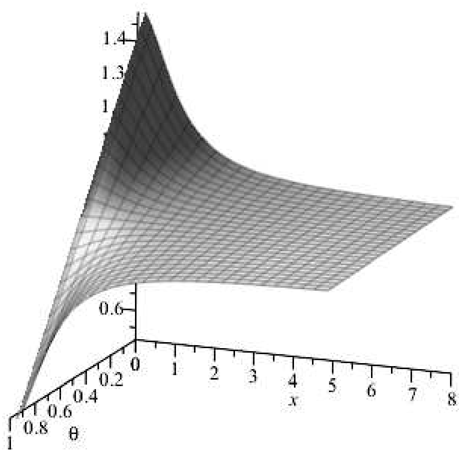

Figure 17: The density of the measure in Configuration 3 when $\gamma = 2$.

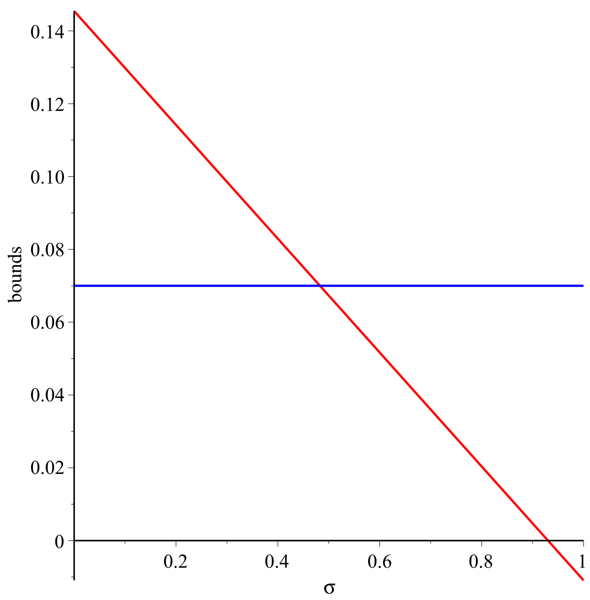

Figure 18: Lower bound (red) and upper bound (blue) in (19) as functions of σ.

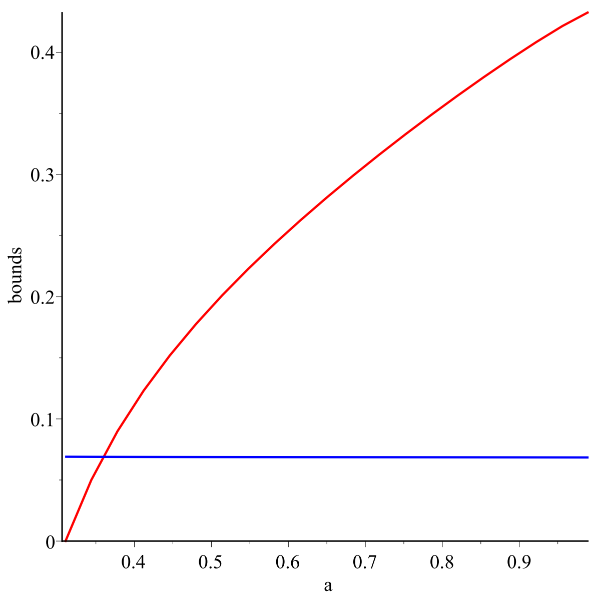

Figure 19: Lower bound (red) and upper bound (blue) in (19) as functions of a.

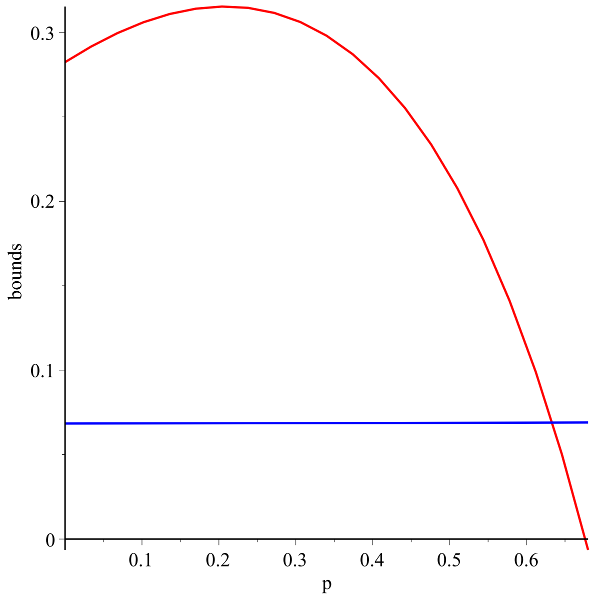

Figure 20: Lower bound (red) and upper bound (blue) in (19) as functions of p.

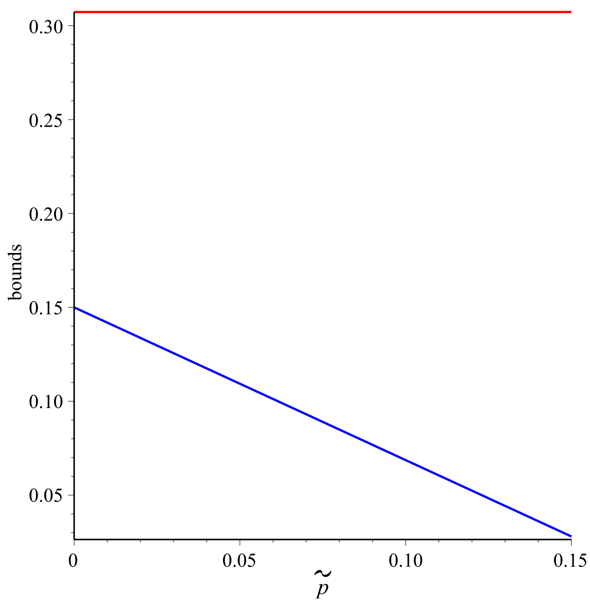

Figure 21: Lower bound (red) and upper bound (blue) in (19) as functions of $\tilde{p}$.

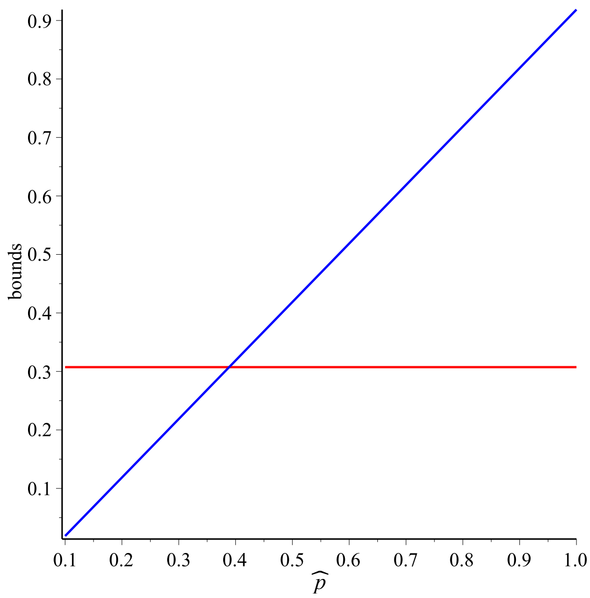

Figure 22: Lower bound (red) and upper bound (blue) in (19) as functions of $\hat{p}$.

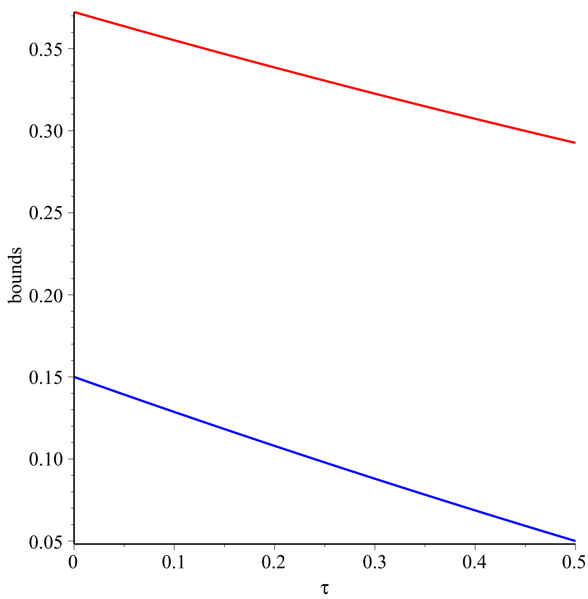

Figure 23: Lower bound (red) and upper bound (blue) in (19) as functions of τ.

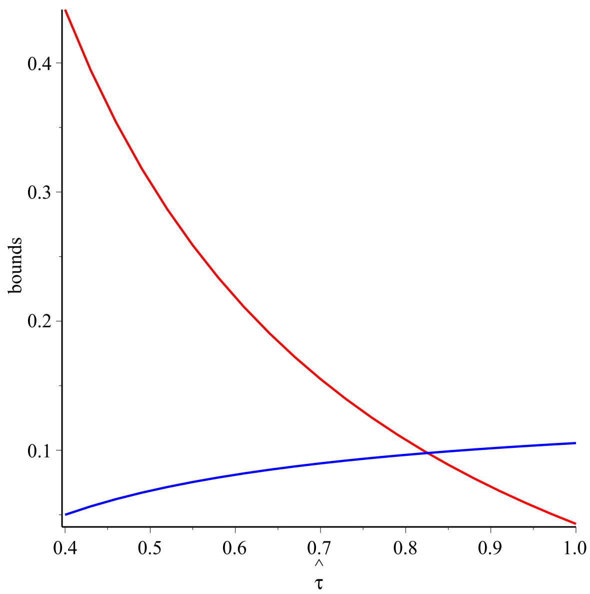

Figure 24: Lower bound (red) and upper bound (blue) in (19) as functions of $\hat{\tau}$.

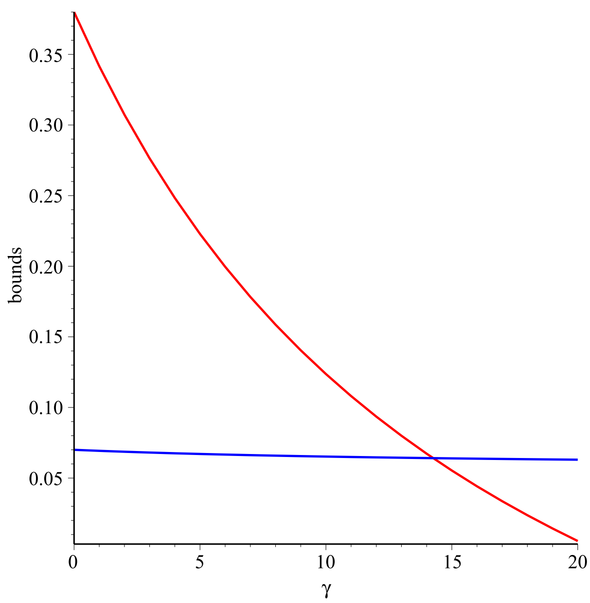

Figure 25: Lower bound (red) and upper bound (blue) in (19) as functions of γ.

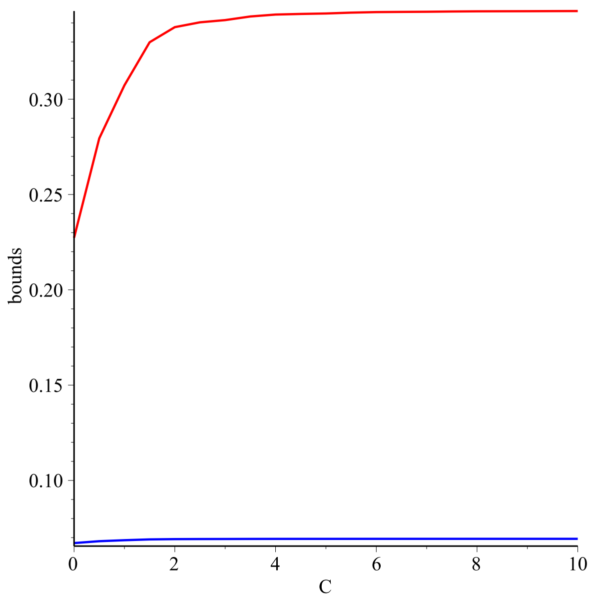

Figure 26: Lower bound (red) and upper bound (blue) in (19) as functions of c.

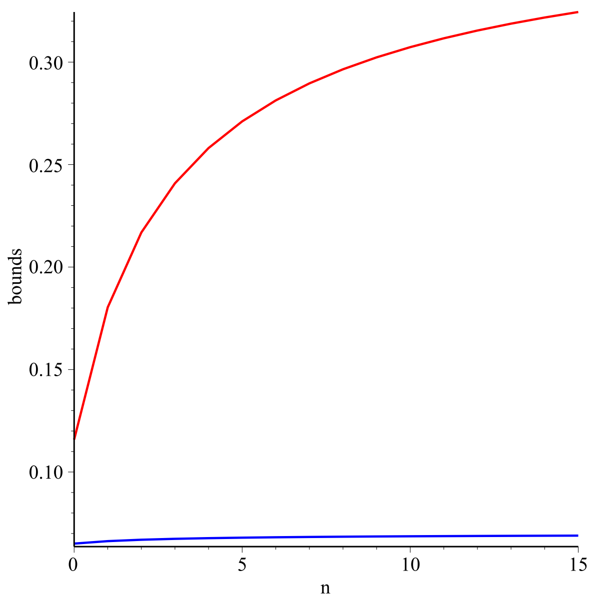

Figure 27: Lower bound (red) and upper bound (blue) in (19) as functions of n.

4 Desirable extensions

One needs to generalize the above model in a variety of directions. These include the following:

1. introduce actors 3, 5 and 6 to obtain a more complete and realistic view of the implications of legalizing some pirate forums;

2. model the increase of the probabilities $(z_k)_{k \leqslant L}$ and deduce the corresponding variations of profits for all actors;

3. it seems natural to assume that all goods are not available at all times from forums. Instead, more popular ones will be more likely to be available. Thus, a more realistic model will attach to each good g_i a probability of being available which is directly proportional to its popularity;

4. consider more general forms for the contribution paid to industry by forums willing to become legal;

5. introduce a regulation authority (the State, or Hadopi), which will benefit from the legalization of forums both in an immaterial way (a society with less crime, more peaceful relations between forums and industry) and in a material one, with added fiscal incomes from industry revenue, or even avoiding extreme negative externalities such as the collapse of parts of the industry if pirating gets out of control;

6. consider in greater details the specific cases of the music and film industries.

References

[1] CURIEN, N. AND MOREAU, F. (2005). The Music Industry in the Digital Era: Towards New Business Frontiers? *Working paper, Laboratoire d'Econométrie, Conservatoire National des Arts et Métiers Paris, February 9.*