Trace and extension results for a class of domains with self-similar boundary

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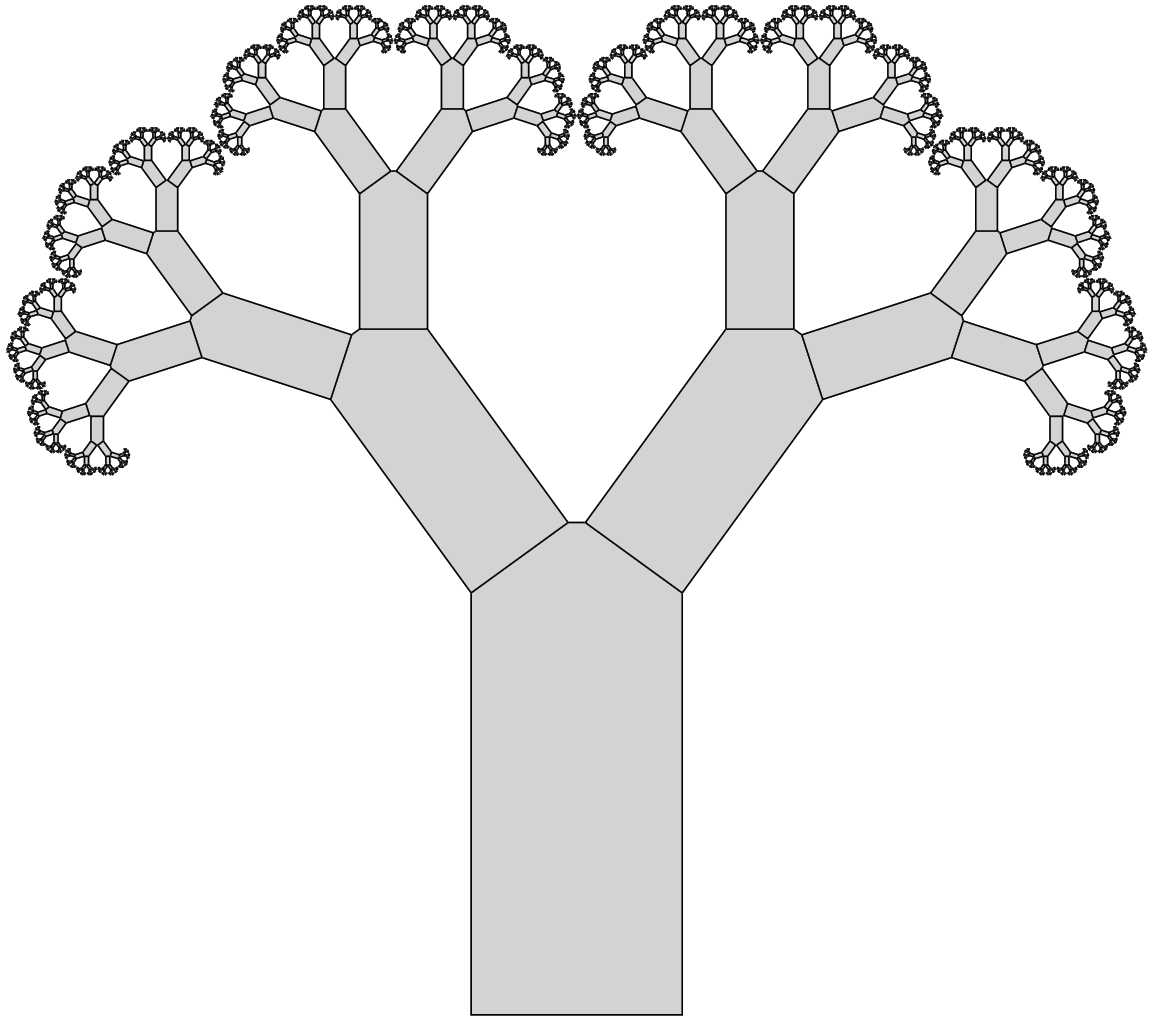
ENS Rennes

Joint work with Yves Achdou and Nicoletta Tchou

May 14 2013

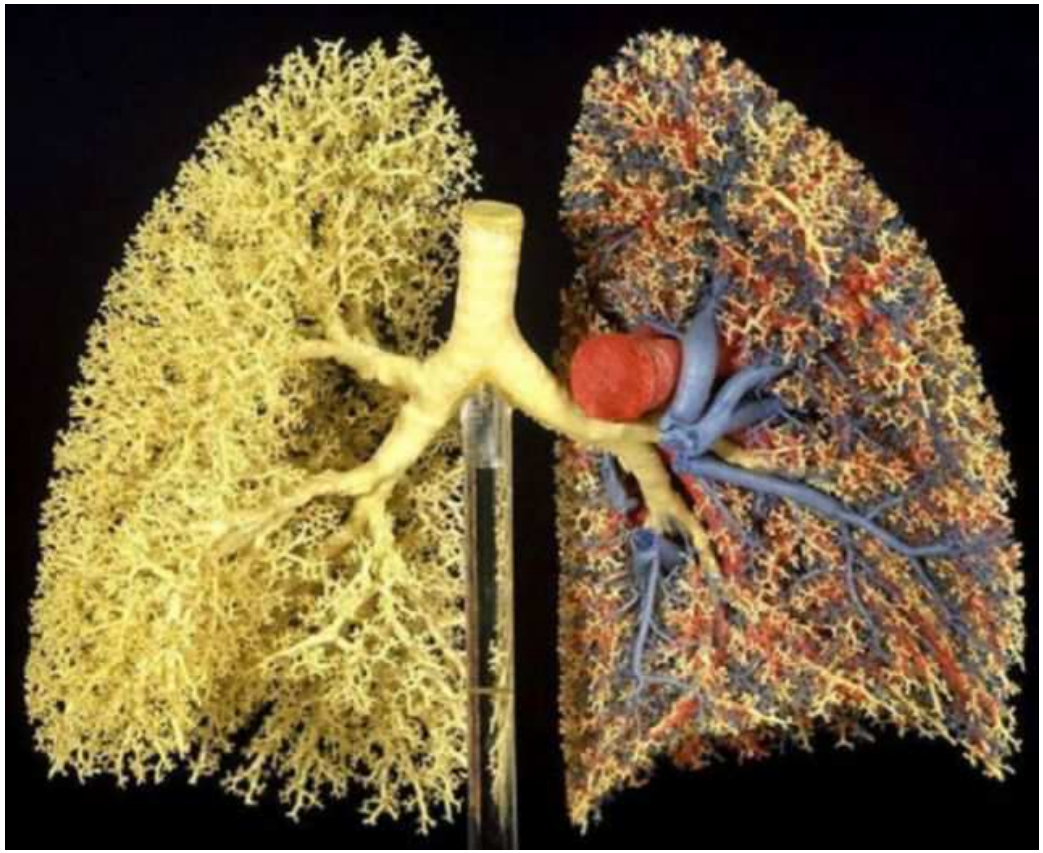
Conference in honour of Kenneth Falconer's 60th birthday

INRIA Saclay

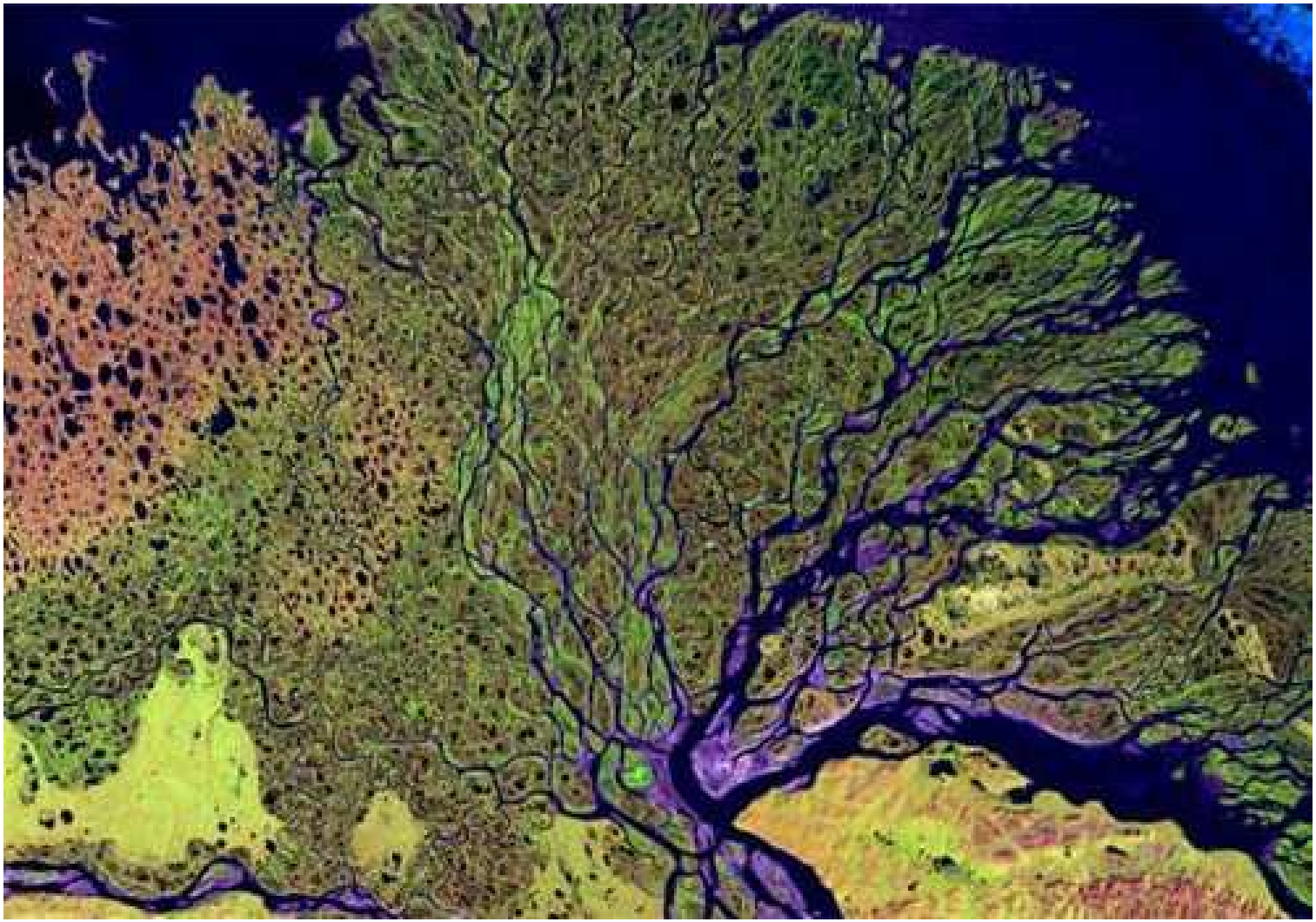


An example of ramified domain

Motivations



Human lungs (E.R. Weibel)



Lena river delta

Outline

1. Presentation
   - The self-similar boundary
   - The class $\mathcal{R}$ of ramified domains
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3. A transmission problem with a fractal interface
   - The transmission problem
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A class of self-similar sets

Consider an iterated function system (f_1, f_2), where

- f_1 and f_2 have ratio $a < 1$,
- f_1 and f_2 have opposite angles $\pm\theta$ $(0 \leqslant \theta < \frac{\pi}{2})$.

Denote Γ the invariant set associated with (f_1, f_2) : $\Gamma = f_1(\Gamma) \cup f_2(\Gamma)$.

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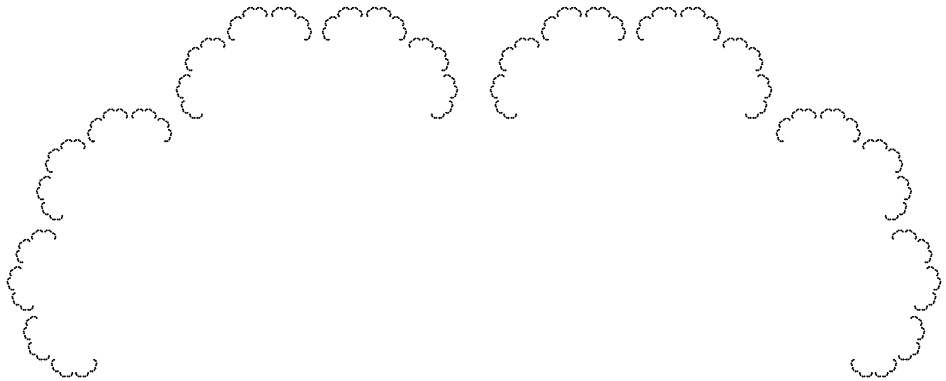

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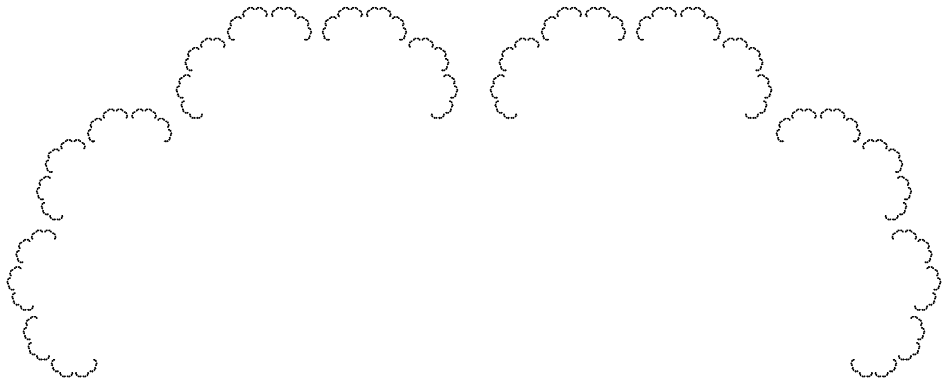

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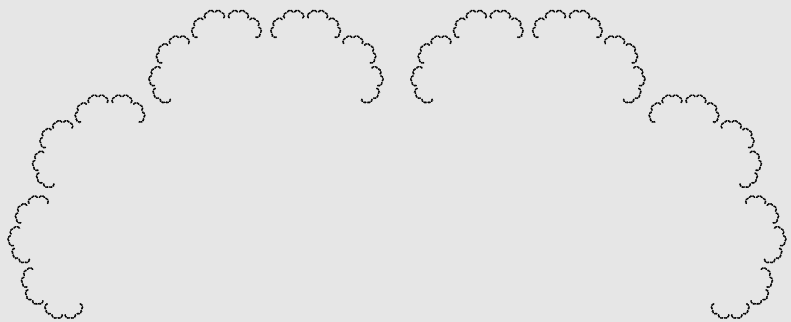
- B. Mandelbrot, M. Frame *The canopy and shortest path in a self-contacting fractal tree*, 1999.

The critical ratio $a^\star$

Proposition

There exists a critical ratio $a^\star$ depending only on the angle θ such that

- *if $a < a^\star$, then Γ is totally disconnected,*



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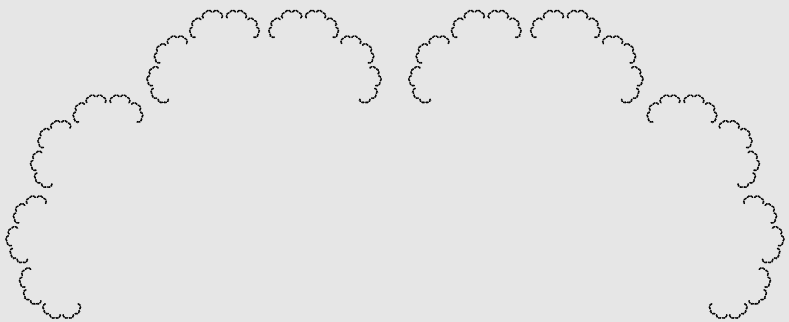
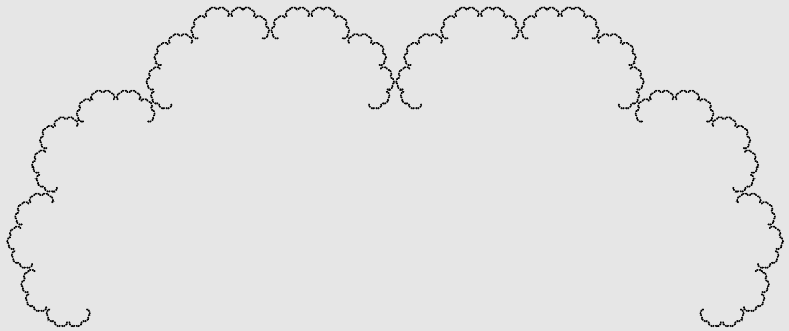


- *if $a = a^\star$, then Γ is connected and has multiple points.*



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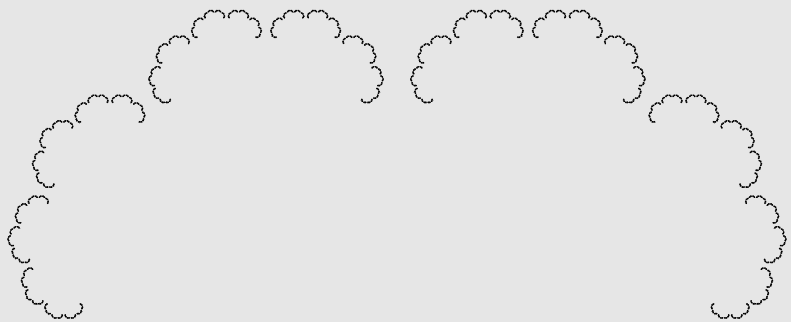

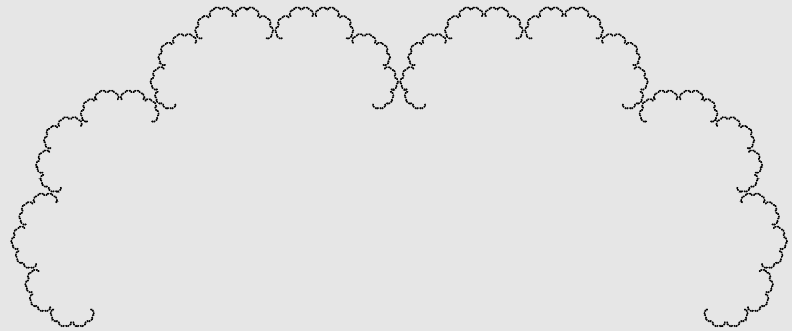

If $a \leqslant a^\star$, Hausdorff dimension of Γ:

$$d := \dim_H(\Gamma) = -\frac{\log 2}{\log a}$$

Sobolev spaces on Γ

The self-similar set Γ is endowed with its invariant measure, *i.e.* the only probability measure μ on Γ satisfying

$$\mu(B) = \frac{1}{2}\mu(f_1^{-1}(B)) + \frac{1}{2}\mu(f_2^{-1}(B))$$

for every Borel set $B \subset \Gamma$.

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The spaces $W^{s,p}(\Gamma)$ (Jonsson, Wallin, 1984)

For $0 < s < 1$ and $1 \leqslant p < \infty$, if $v \in L^p_\mu(\Gamma)$, then $v \in W^{s,p}(\Gamma)$ if and only if

$$|v|^p_{W^{s,p}(\Gamma)} \;:=\; \iint_{|x-y|<1} \frac{|v(x)-v(y)|^p}{|x-y|^{d+ps}}\, \mathrm{d}\mu(x)\, \mathrm{d}\mu(y) \;<\; \infty.$$

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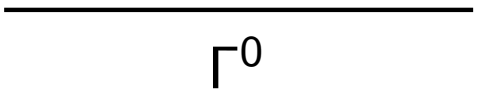
The class $\mathcal{R}$ of ramified domains



Γ^0

First cell Y^0

The class $\mathcal{R}$ of ramified domains

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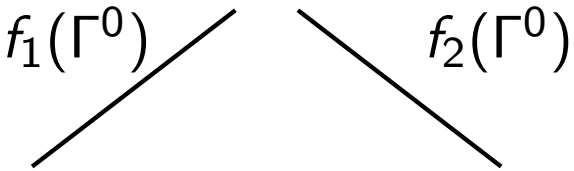

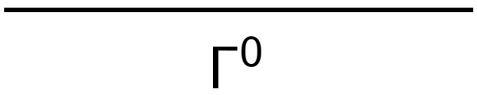

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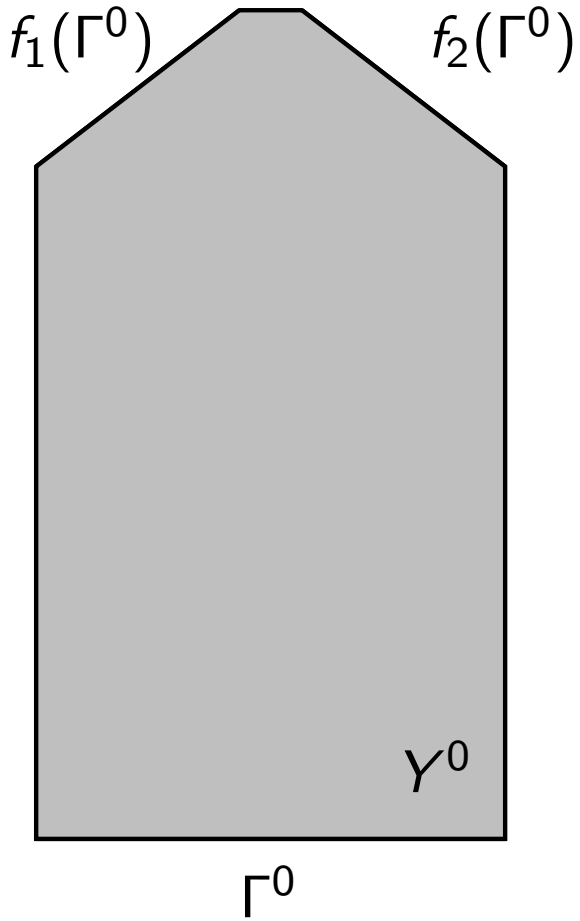

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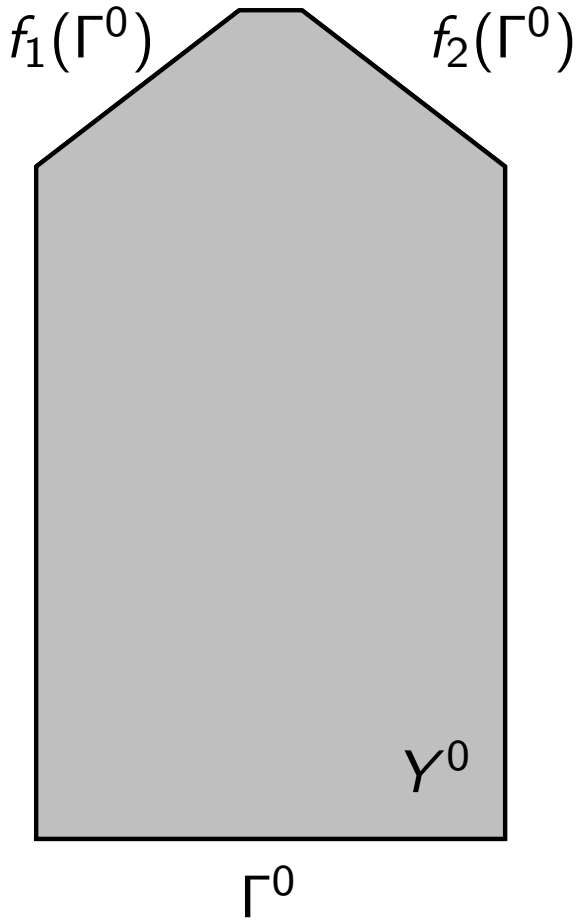

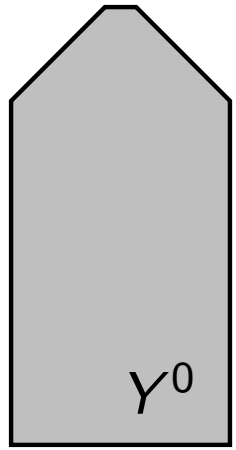

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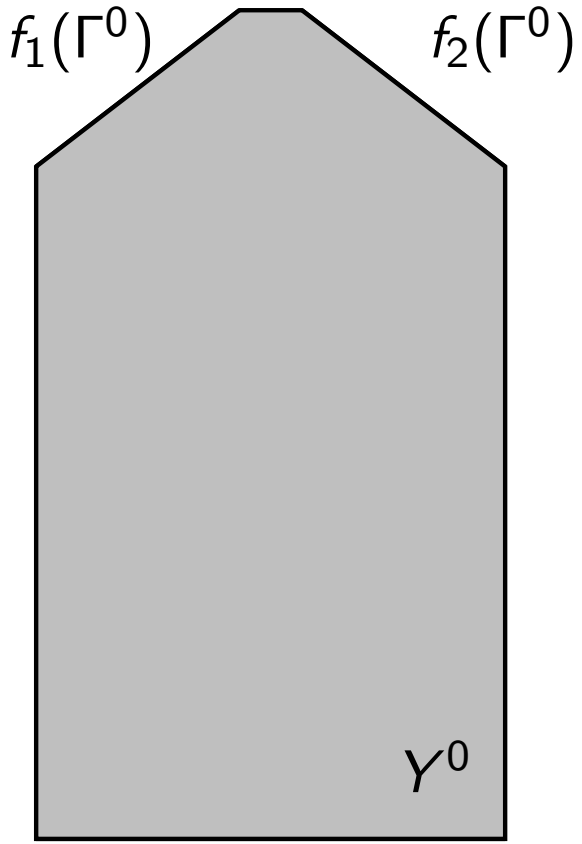
The class $\mathcal{R}$ of ramified domains



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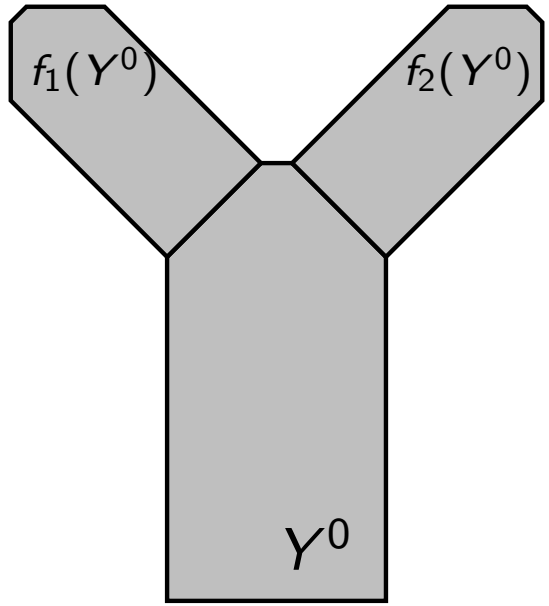


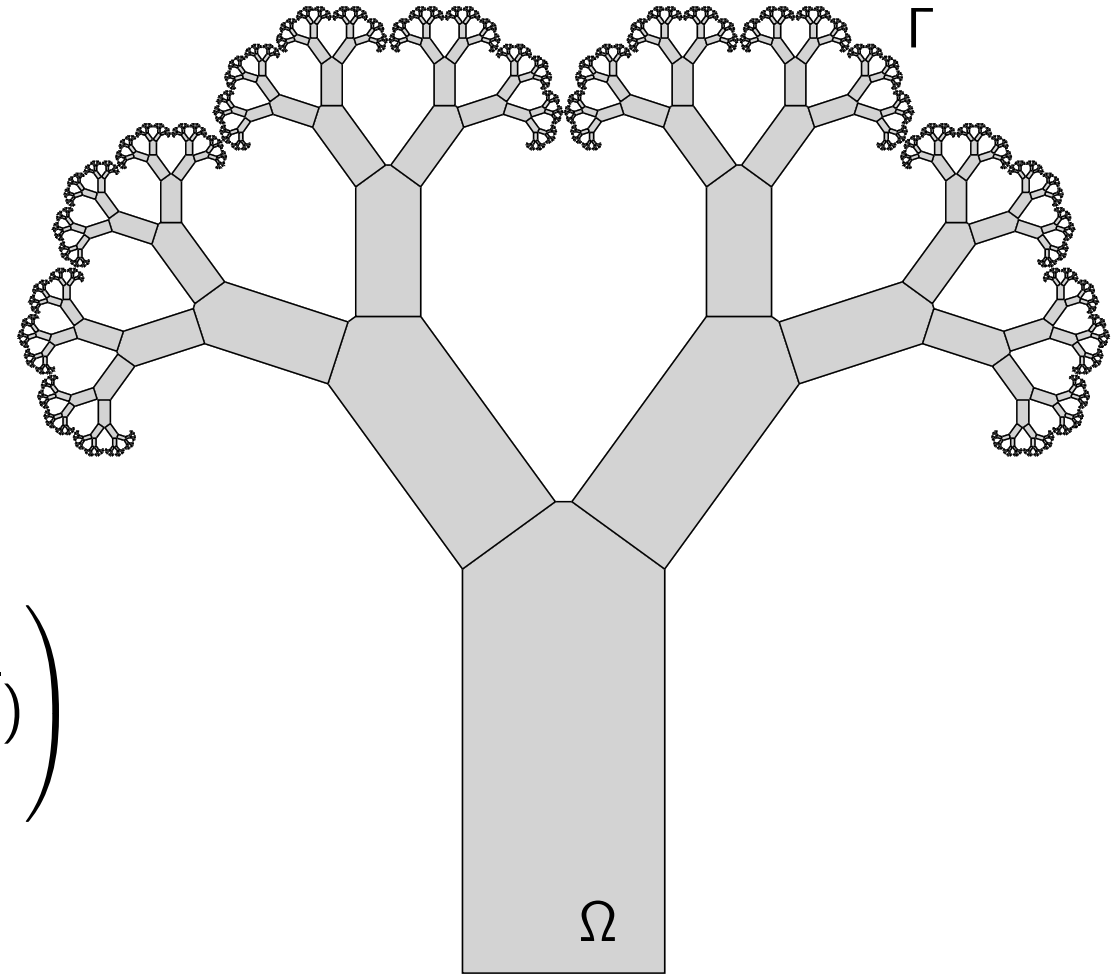
Second iteration

The class $\mathcal{R}$ of ramified domains

If $\sigma = (\sigma_1, \ldots, \sigma_k) \in \{1,2\}^k$, write

$$f_\sigma := f_{\sigma_1} \circ \ldots \circ f_{\sigma_k}.$$

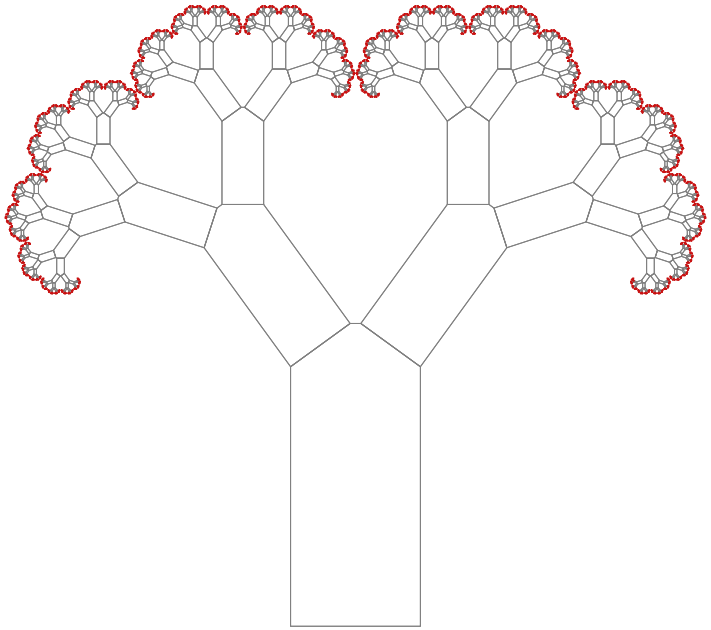
$$\Omega = \text{Interior}\left(\bigcup_{k \geqslant 0} \bigcup_{\sigma \in \{1,2\}^k} f_\sigma(\overline{Y^0}) \right)$$



The set Ξ of multiple points of Γ

$a = a^\star$

First geometry



$$\theta \notin \{\tfrac{\pi}{2k},\ k \in \mathbb{N}^\star\}$$

Ξ is countable

where Ξ is the set of multiple points of Γ.

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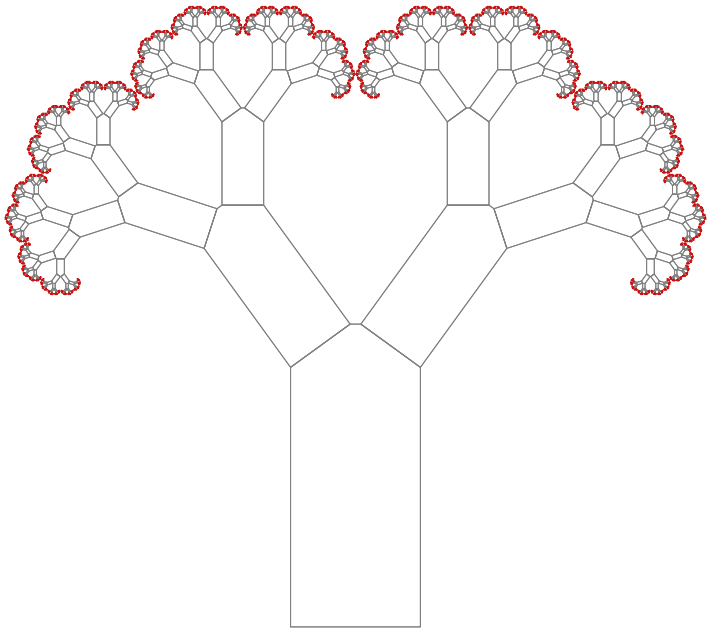

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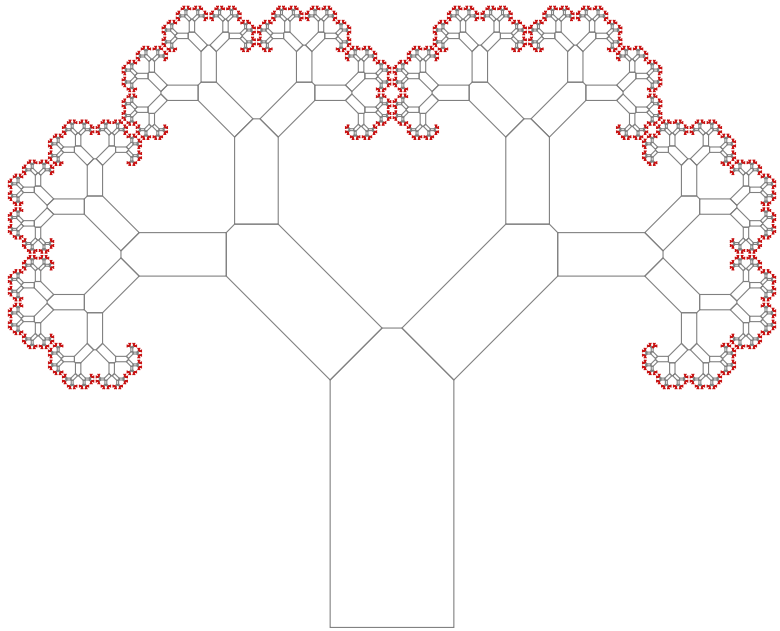

$\theta = \frac{\pi}{2k},\ k \in \mathbb{N}^\star$

$\dim_H \Xi = \frac{d}{2}$

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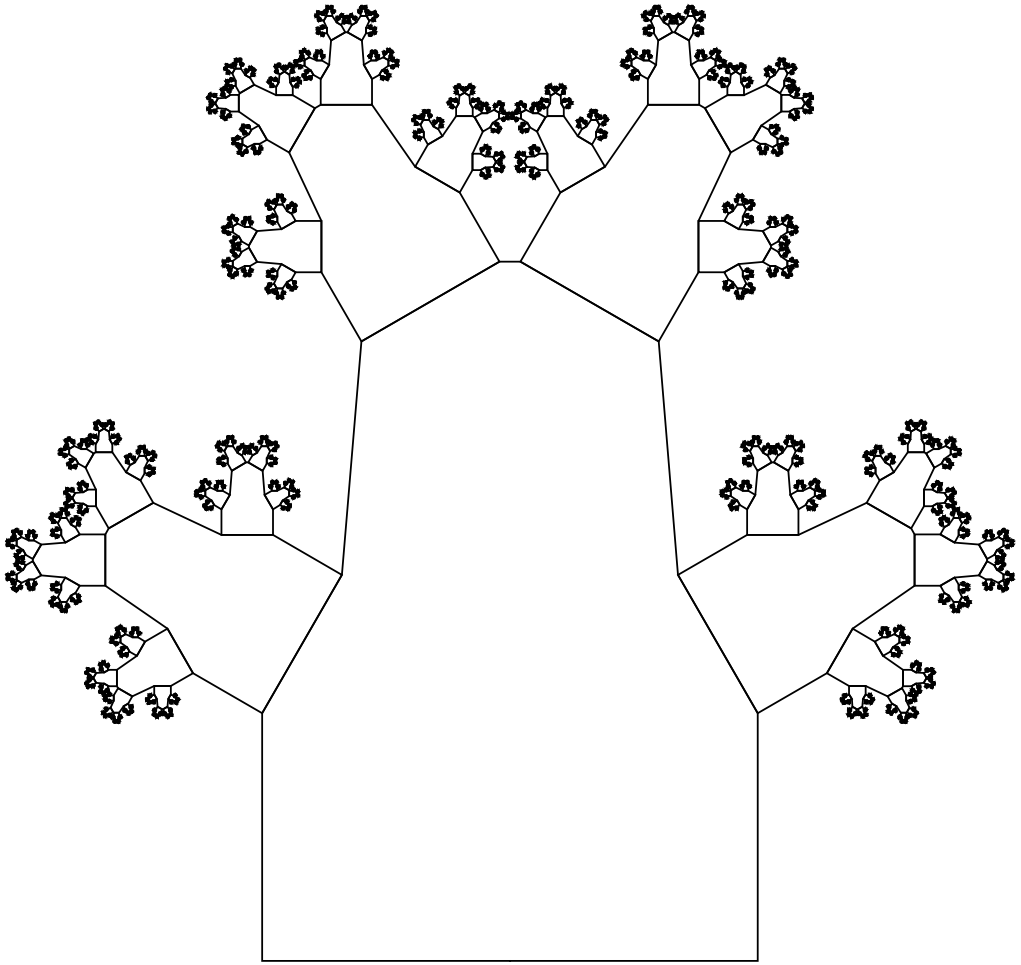
Another example

A case with 4 similitudes : angles $\pm\frac{\pi}{3}$, $\pm\frac{\pi}{6}$.



Self-similar set:

$$\Gamma = \bigcup_{i=1}^{4} f_i(\Gamma)$$

$$\dim_H \Xi = \frac{\dim_H \Gamma}{4}$$

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Questions

- **Trace**:
  - Notion of trace for functions in $W^{1,p}(\Omega)$ on Γ ?
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  - for which values of p is the domain Ω a $W^{1,p}$-extension domain, *i.e.* there exists a linear continuous extension operator

$$W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)?$$

If Ω is a $W^{1,p}$-extension domain for every $p \in [1, \infty]$, Ω is said to be a Sobolev extension domain.

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Whitney (1934), Stein (1970), Jones (1981), Hajłasz, Koskela *et al.* (1998-)

Lipschitz domains

- **Trace theorem**

Theorem (Gagliardo, 1957)

If $\omega \subset \mathbb{R}^n$ is an open set with Lipschitz boundary and $1 < p < \infty$, one has the trace result:

$$W^{1,p}(\omega)_{|\partial\omega} \;=\; W^{1-\frac{1}{p},p}(\partial\omega).$$

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- **Extension theorem**

Theorem (A.P. Calderón, E.M. Stein, 1970)

Every domain $\omega \subset \mathbb{R}^n$ with Lipschitz boundary is a Sobolev extension domain.

Jones domains

Definition (Jones domains, P.W. Jones, 1981)

A domain $\omega \subset \mathbb{R}^n$ is called a Jones domain if there exists $\varepsilon, \delta > 0$ such that for every $x, y \in \omega$ satisfying $|x - y| < \delta$, there exists a rectifiable arc $\gamma \subset \omega$ joining x to y such that

- $L(\gamma) \leqslant \frac{1}{\varepsilon} |x - y|$ *where* $L(\gamma) =$*length of* γ,

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- P.W. Jones, *Quasiconformal mappings and extendability of functions in Sobolev spaces*, 1981.
- O. Martio, J. Sarvas, *Injectivity theorems in plane and space*, 1979.

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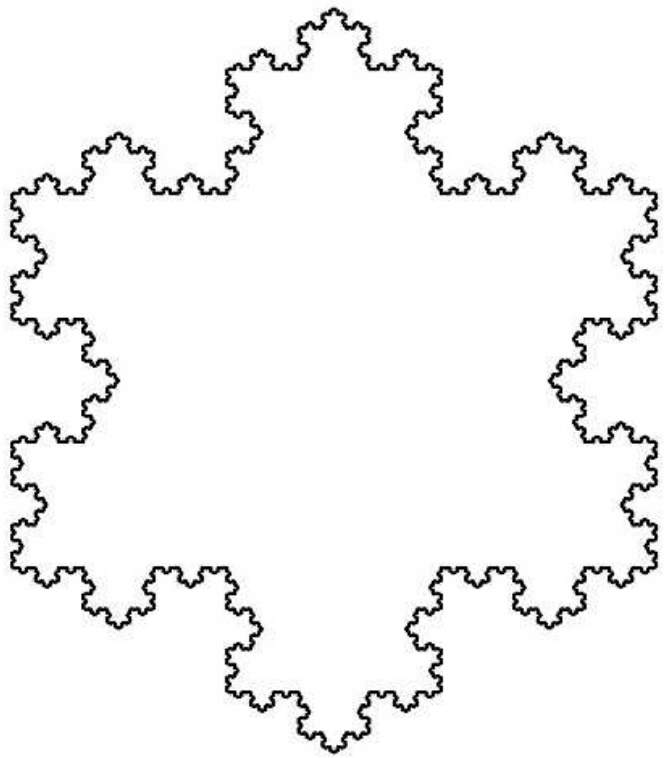
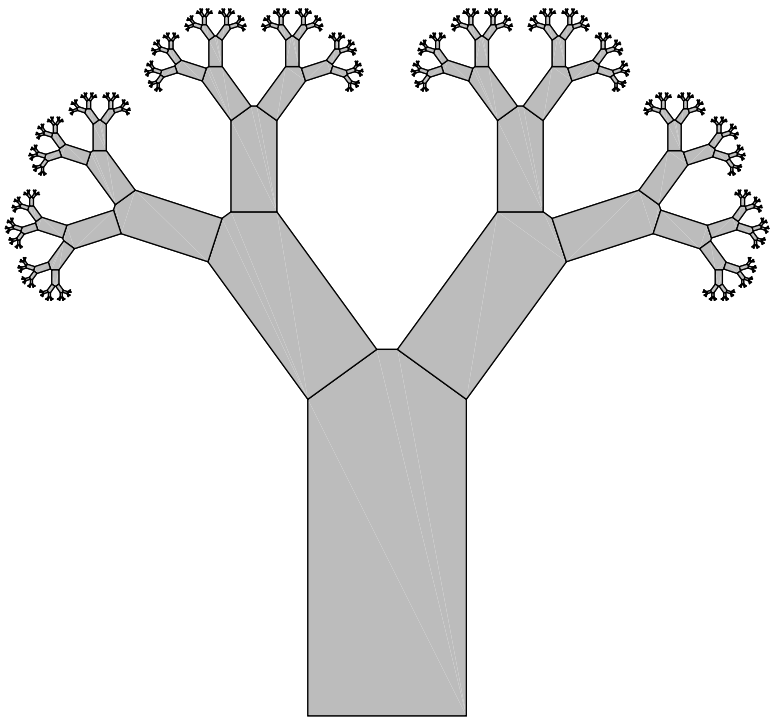
An example of Jones domain : the Koch flake



The subcritical domains of the class $\mathcal{R}$ $(a < a^\star)$

- **Extension**



Proposition

If Ω is a ramified domain of the class $\mathcal{R}$with $a < a^\star$, then Ω is a Jones domain.

Therefore, Ω is a Sobolev extension domain.

The subcritical domains of the class $\mathcal{R}$ $(a < a^\star)$

- **Traces**

Theorem (A. Jonsson, H. Wallin, 1984)

If $1 < p < \infty$, *and* $1 - \frac{2-d}{p} > 0$, *then*

$$W^{1,p}(\mathbb{R}^2)_{|\Gamma} = W^{1-\frac{2-d}{p},p}(\Gamma)$$

where $d = \dim_H \Gamma$.

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where $d = \dim_H \Gamma$.

Sense of the trace: the function u is strictly defined at $x \in \mathbb{R}^2$ if the limit

$$\overline{u}(x) \;:=\; \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\,\mathrm{d}y$$

exists. Define the trace of u on Γ to be the function $\overline{u}_{|\Gamma}$, defined at the points where u is strictly defined.

The subcritical domains of the class $\mathcal{R}$ $(a < a^\star)$

For $p \in (1, \infty)$, we deduce from Jones theorem that there exists an extension operator

$$W^{1,p}(\Omega) \ \to \ W^{1,p}(\mathbb{R}^2).$$

Combining with Jonsson and Wallin's trace operator, we get a trace operator

$$W^{1,p}(\Omega) \ \to \ W^{1-\frac{2-d}{p},p}(\Gamma).$$

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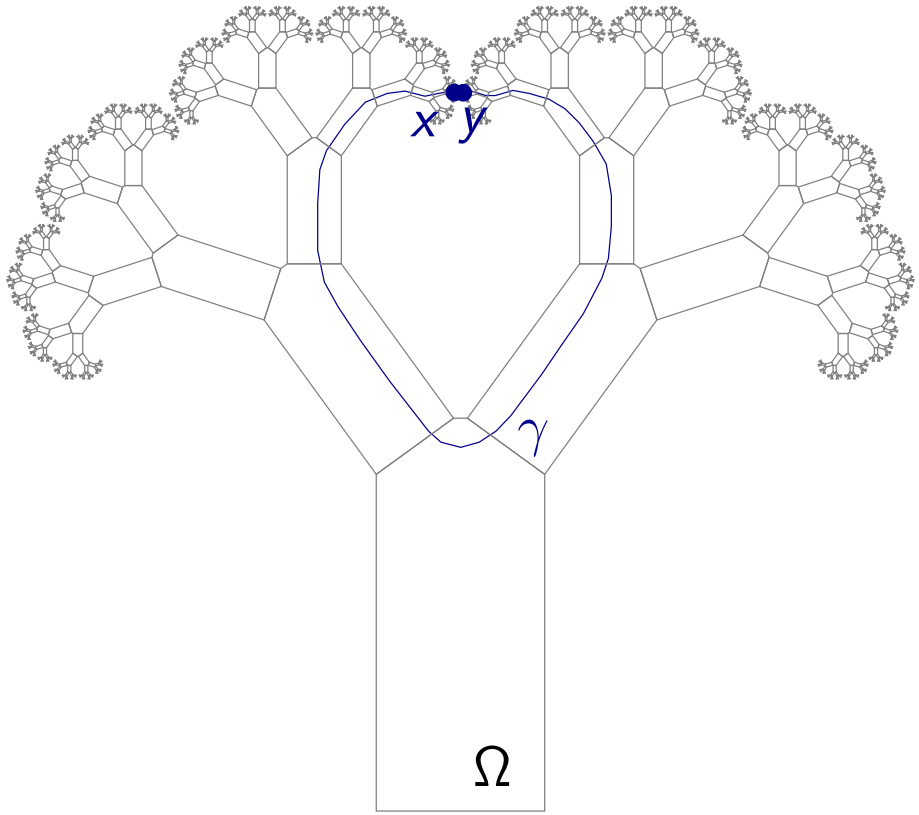
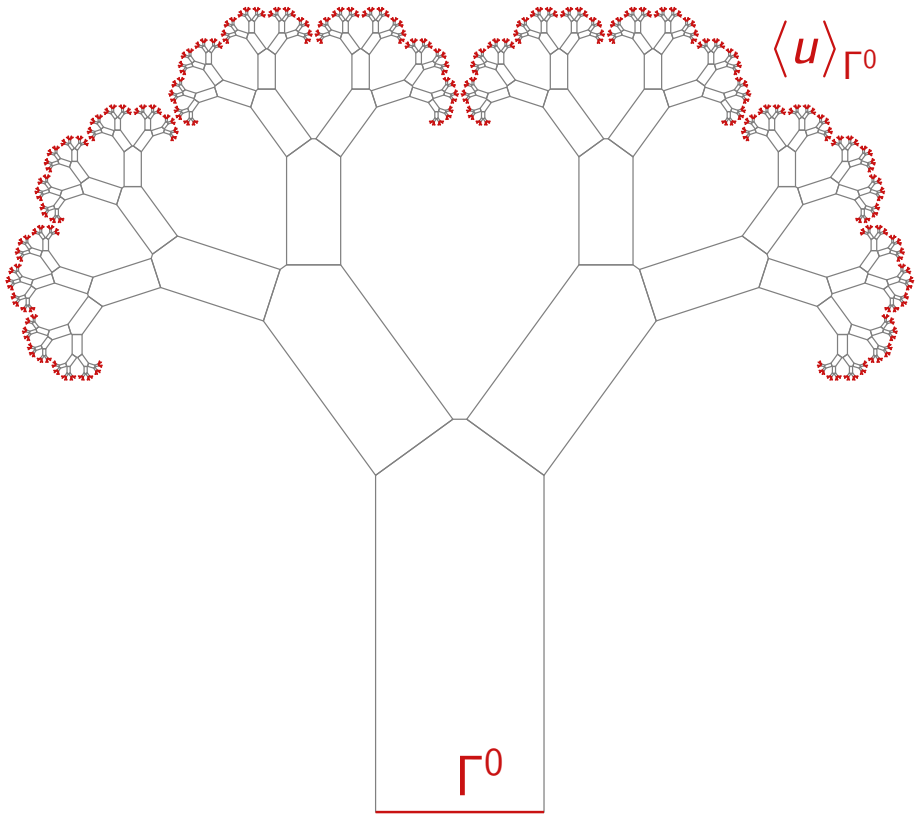
The ramified domains of $\mathcal{R}$ with $a = a^\star$

The case $a = a^\star$



In this case, Ω cannot be a $W^{1,p}$-extension domain for $p > 2$.

Construction of a self-similar trace operator



For $u \in W^{1,p}(\Omega)$, define:

$$\ell^0(u) \equiv \langle u \rangle_{\Gamma^0},$$

where $\langle u \rangle_{\Gamma^\sigma} = \dfrac{1}{|\Gamma^\sigma|} \displaystyle\int_{\Gamma^\sigma} u.$

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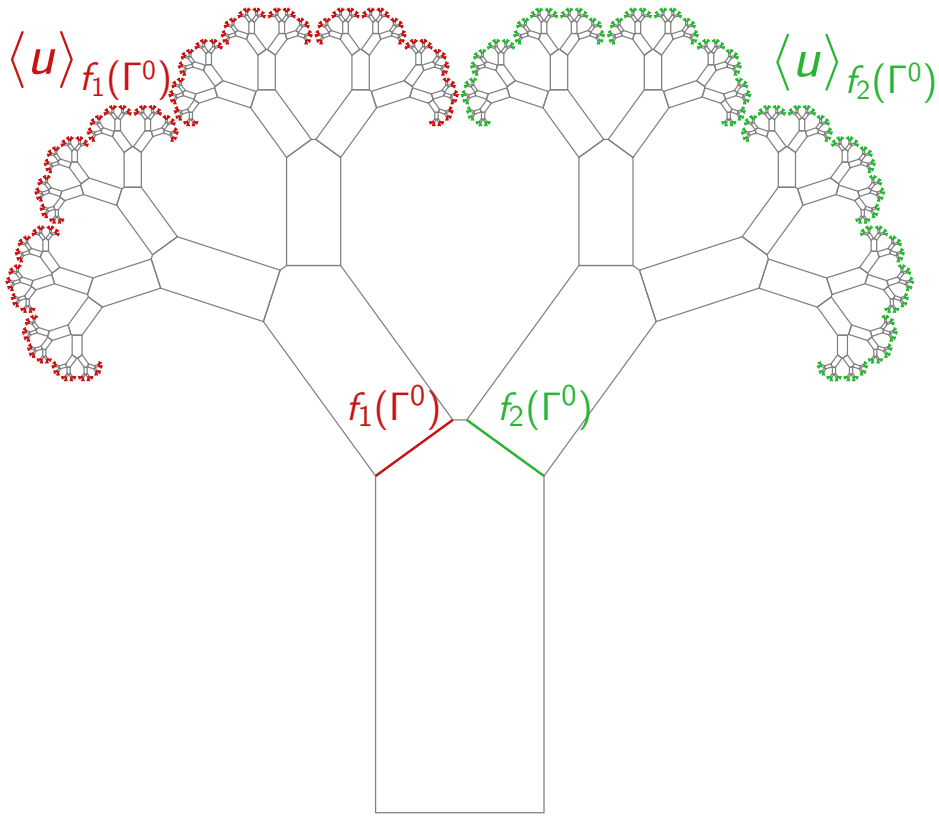
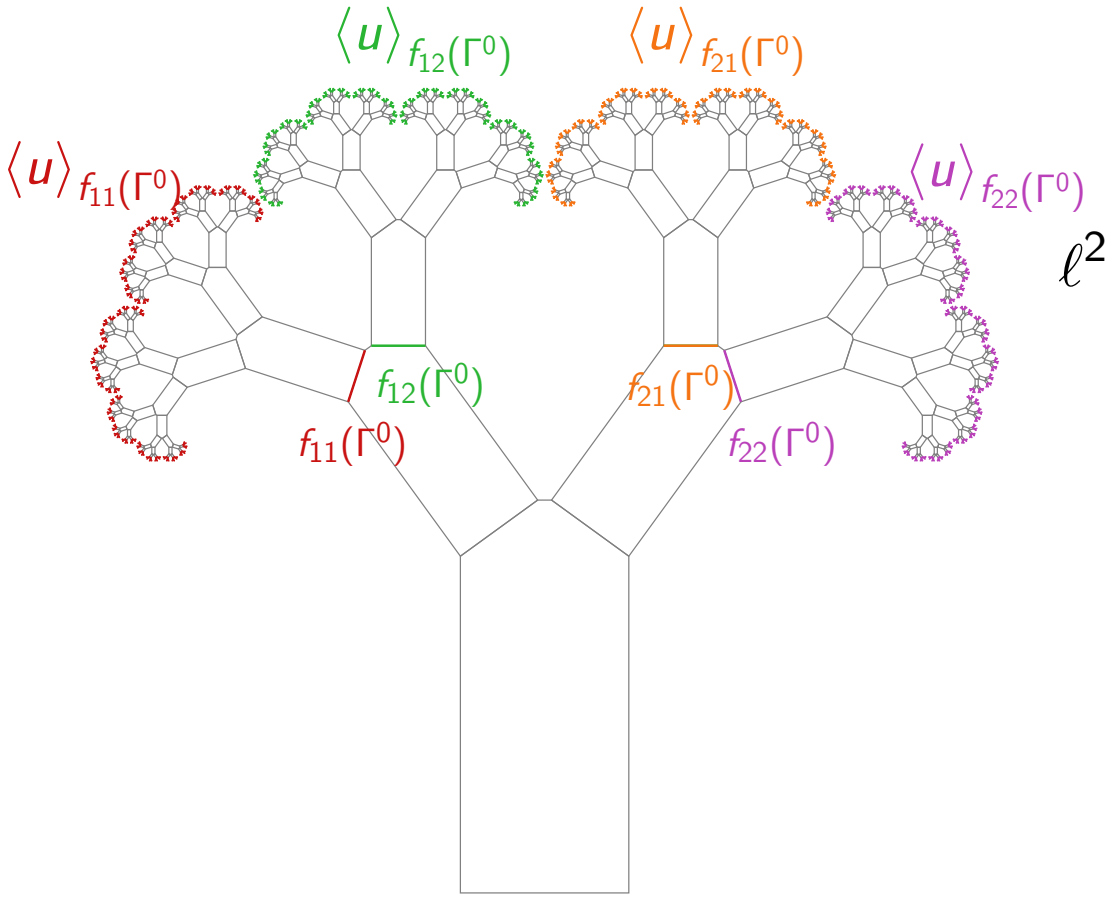


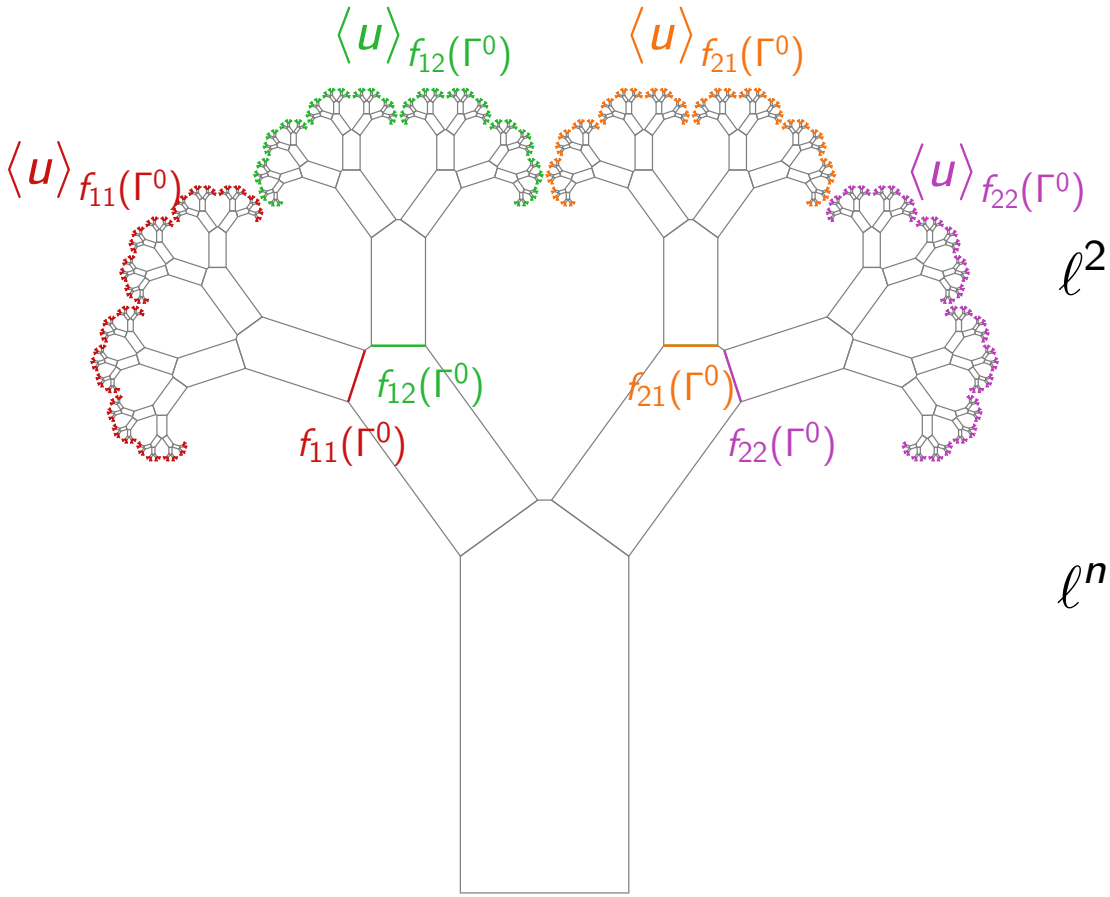
$$\ell^1(u) = \langle u \rangle_{f_1(\Gamma^0)} \mathbb{1}_{f_1(\Gamma)} + \langle u \rangle_{f_2(\Gamma^0)} \mathbb{1}_{f_2(\Gamma)}$$

Construction of a self-similar trace operator



$$\ell^2(u) = \sum_{\sigma \in \{1,2\}^2} \langle u \rangle_{f_\sigma(\Gamma^0)} \mathbb{1}_{f_\sigma(\Gamma)}$$

Construction of a self-similar trace operator



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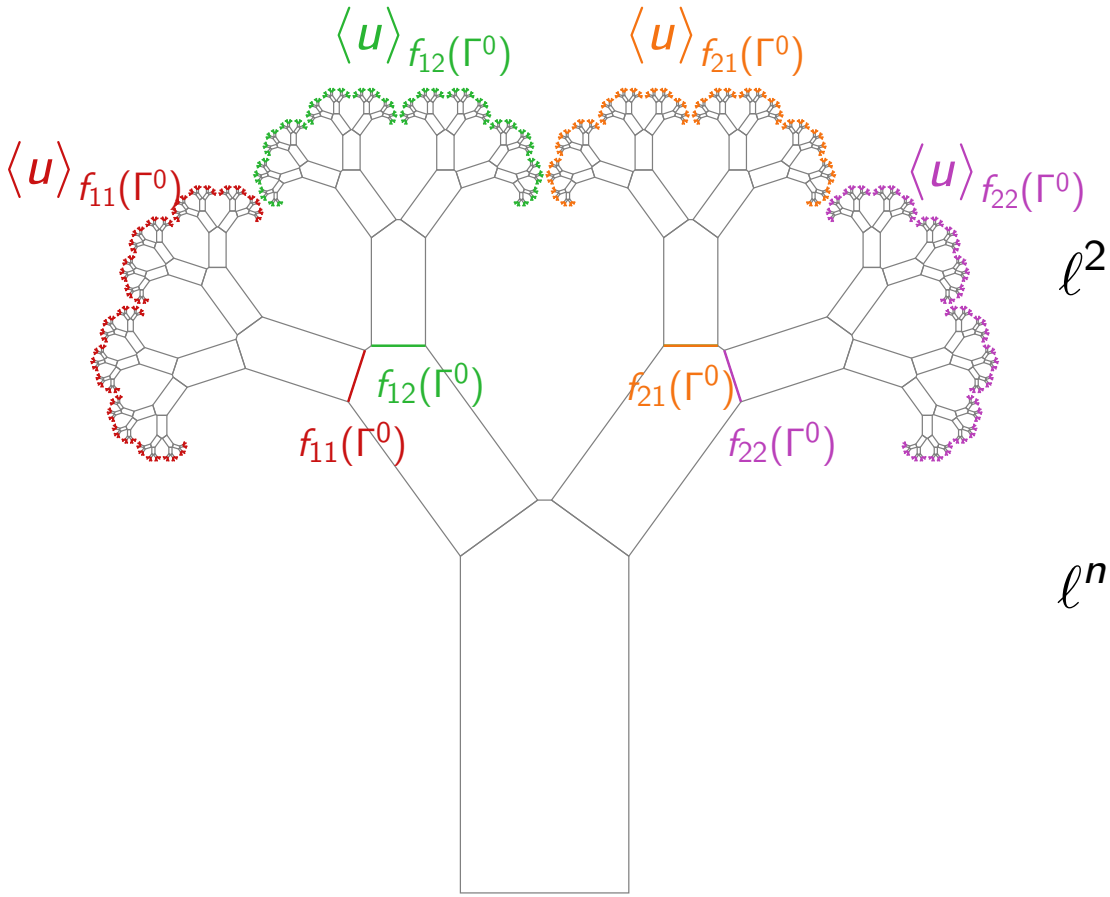

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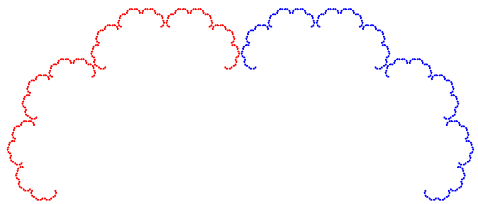
Proposition

The sequence (ℓ^n) converges in $\mathcal{L}(W^{1,p}(\Omega), L^p_\mu(\Gamma))$ to an operator ℓ^∞.

Characterization of the trace space on Γ

Haar wavelets on Γ:

$$\begin{cases} g_0 = \mathbb{1}_{f_1(\Gamma)} - \mathbb{1}_{f_2(\Gamma)} \end{cases}$$

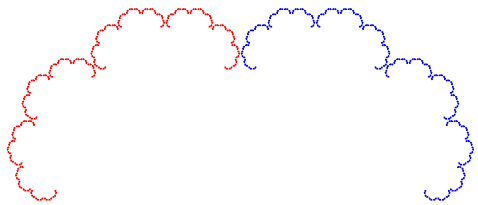


Mother wavelet g_0

Characterization of the trace space on Γ

Haar wavelets on Γ:

$$\begin{cases} g_0 & = & \mathbb{1}_{f_1(\Gamma)} - \mathbb{1}_{f_2(\Gamma)} \\ g_{\sigma|f_\sigma(\Gamma)} & = & 2^{\frac{k}{2}} g_0 \circ {f_\sigma}^{-1} \text{ and } g_{\sigma|\Gamma\setminus f_\sigma(\Gamma)} = 0 \text{ for } \sigma \in \{1,2\}^k \end{cases}$$



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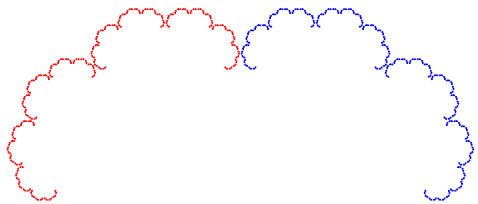

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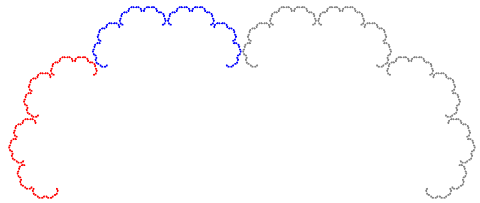
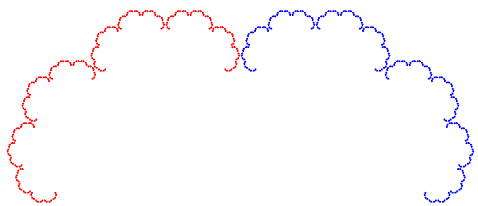


g_σ for $\sigma = (1)$

Characterization of the trace space on Γ

Haar wavelets on Γ:

$$\begin{cases} g_0 &= \mathbb{1}_{f_1(\Gamma)} - \mathbb{1}_{f_2(\Gamma)} \\ g_{\sigma\,|f_\sigma(\Gamma)} &= 2^{\frac{k}{2}} g_0 \circ {f_\sigma}^{-1} \quad \text{and} \quad g_{\sigma\,|\Gamma\setminus f_\sigma(\Gamma)} = 0 \quad \text{for} \quad \sigma \in \{1,2\}^k \end{cases}$$



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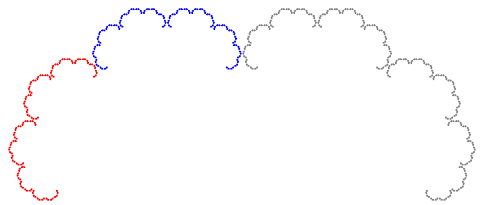

g_σ for $\sigma = (1)$

Every function $v \in L^p_\mu(\Gamma)$, $1 \leqslant p < \infty$ can be expanded in the Haar wavelet basis (g_σ):

$$v \;=\; \langle v \rangle_\Gamma + \sum_{k \geqslant 0} \sum_{\sigma \in \{1,2\}^k} \beta_\sigma g_\sigma .$$

Characterization of the trace space on Γ

Theorem (Y. Achdou, N. Tchou, 2010)

If $1 < p < \infty$, *then*

$$\ell^\infty(W^{1,p}(\Omega)) \;=\; JLip^{1-\frac{2-d}{p},p}(\Gamma).$$

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Definition (The spaces *JLip*, A. Jonsson, 2004)

The space $JLip^{s,p}(\Gamma)$ *is made of the functions* $v \in L^p_\mu(\Gamma)$ *such that*

$$\sum_{k\geqslant 0} 2^{\frac{ksp}{d}} 2^{k\left(\frac{p}{2}-1\right)} \sum_{\sigma\in\{1,2\}^k} |\beta_\sigma|^p \;<\; \infty,$$

where the β_σ *are the wavelet coefficients of* v.

The spaces $JLip^{s,p}(\Gamma)$ authorize jumps at the multiple points of Γ.

- A. Jonsson, *Haar wavelets of higher order on fractals and regularity of functions*, 2004.

Theorem (Y. Achdou, T.D., N. Tchou, 2012)

Set

$$p^\star = 2 - \dim_H \Xi$$

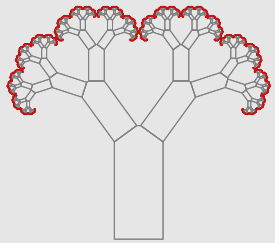
If $a = a^\star$, *then*

- *if* $p < p^\star$,

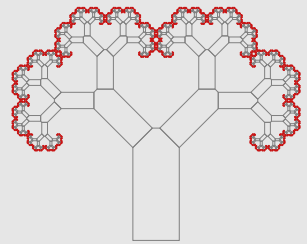
$$\ell^\infty(W^{1,p}(\Omega)) \;=\; W^{1-\frac{2-d}{p},p}(\Gamma)$$

- *if* $p \geqslant p^\star$,

$$\ell^\infty(W^{1,p}(\Omega)) \;\subset\; W^{s,p}(\Gamma) \text{ for every } s < \frac{d-\dim_H \Xi}{p}$$



$p^\star = 2$



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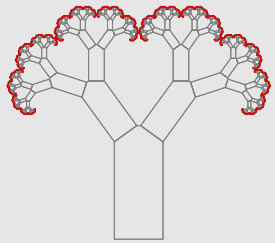

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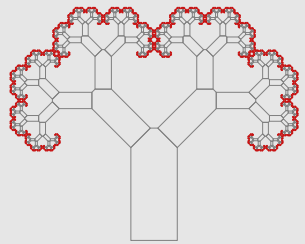
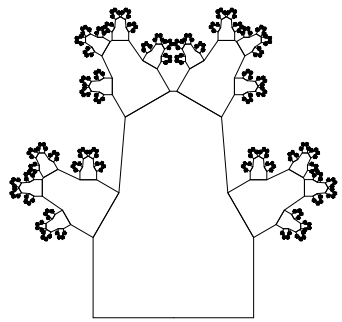


$p^\star = 2 - \frac{d}{2}$

In the case of the ramified domain with 4 similitudes, the result holds.



Trace theorem: ideas of the proof

Regularity of the Haar wavelets

- In the case when $\theta \neq \frac{\pi}{2k}$, $k \in \mathbb{N}^\star$,

$$g_0 \in W^{s,p}(\Gamma) \quad \text{si } s < \frac{d}{p},$$

$$g_0 \notin W^{s,p}(\Gamma) \quad \text{si } s > \frac{d}{p}.$$

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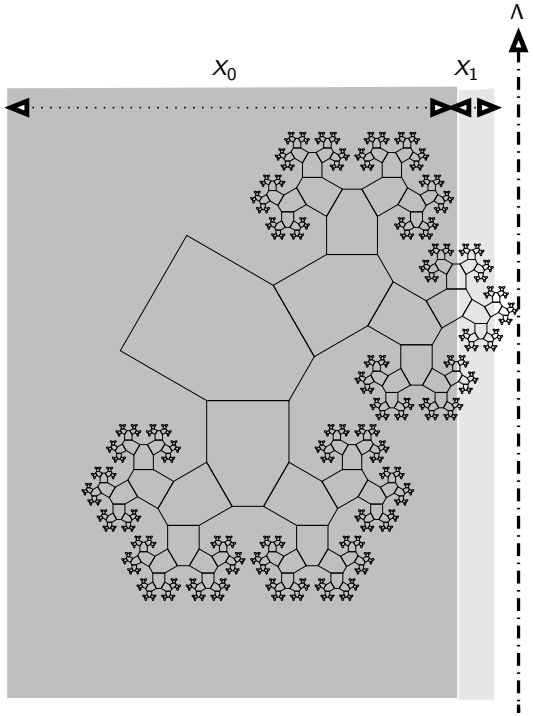
Trace theorem: ideas of the proof

Estimation of the $W^{s,p}$-norm of a function $v \in JLip^{t,p}(\Gamma)$ using:

- a partition of Γ and its images by the similitudes f_σ
- the expansion of v in the Haar wavelet basis
- the discrete Hardy inequalities: $\forall \gamma \in \mathbb{R}$, $\forall p \geqslant 1$, $\exists C > 0$ such that for all sequence of real numbers $(c_k)_{k\in\mathbb{N}}$



$$\sum_n 2^{\gamma n}\left(\sum_{k\leqslant n} c_k\right)^p \leqslant C\sum_n 2^{\gamma n} {c_n}^p \quad \text{if } \gamma < 0,$$

$$\sum_n 2^{\gamma n}\left(\sum_{k\geqslant n} c_k\right)^p \leqslant C\sum_n 2^{\gamma n} {c_n}^p \quad \text{if } \gamma > 0$$

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Extension theorem

In the case $a = a^\star$,

- we know that a ramified domain Ω of $\mathcal{R}$ is not a $W^{1,p}$-extension domain for $p > 2$.
- the trace theorem suggests that Ω is not a $W^{1,p}$-extension domain for $p > p^\star$.
- $p < p^\star$?

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- the trace theorem suggests that Ω is not a $W^{1,p}$-extension domain for $p > p^\star$.
- $p < p^\star$?

Theorem (T.D., 2013)

If Ω is a critical ramified domain of $\mathcal{R}$, and $p^\star = 2 - \dim \Xi$, then for all $p < p^\star$, Ω is a $W^{1,p}$-extension domain

Extension theorem: ideas of the proof

Step 1

Theorem (T.D., 2013)

If $p < p^\star$, there exists a linear continuous lifting operator

$$\mathcal{E} : JLip^{1-\frac{2-d}{p},p}(\Gamma) \;=\; W^{1-\frac{2-d}{p},p}(\Gamma) \to W^{1,p}(\mathbb{R}^2)$$

where the trace operator is ℓ^∞, i.e. $\ell^\infty((\mathcal{E}v)_{|\Omega}) = v$ for all v.

Extension theorem: ideas of the proof

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where the trace operator is ℓ^∞, i.e. $\ell^\infty((\mathcal{E}v)_{|\Omega}) = v$ for all v.

We start by constructing liftings $\tilde{g}_\sigma$ for the Haar wavelets g_σ on Γ.

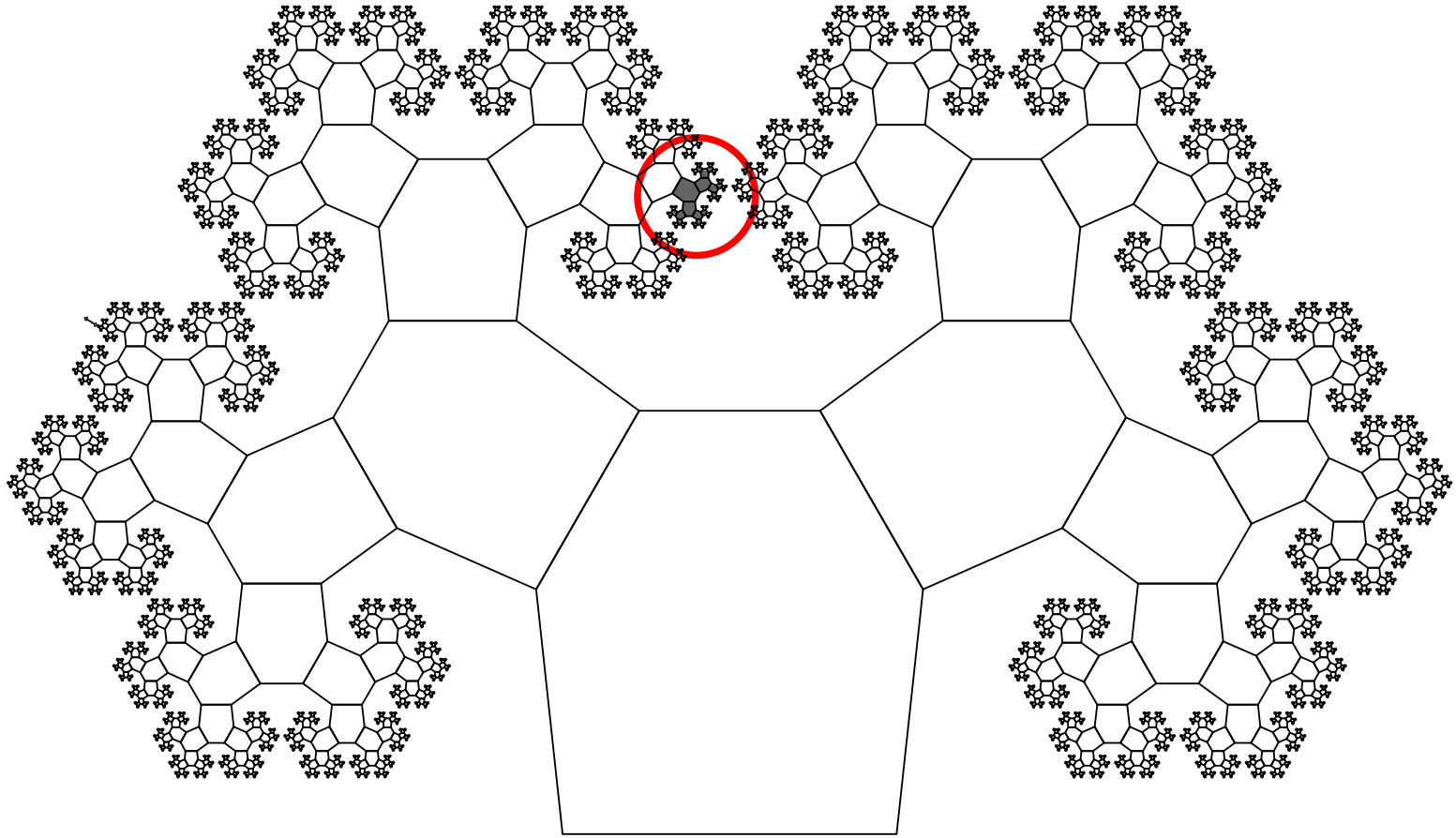
We expand $v \in W^{1-\frac{2-d}{p},p}(\Gamma)$ in the Haar wavelet basis:

$$v \;=\; \langle \ell^\infty(u) \rangle + \sum_{k \geqslant 0} \sum_{\sigma \in \{1,2\}^k} \beta_\sigma g_\sigma$$

and we put $\mathcal{E}v \;=\; \mathcal{E}\langle v \rangle + \sum_{k \geqslant 0} \sum_{\sigma \in \{1,2\}^k} \beta_\sigma \tilde{g}_\sigma.$

Extension theorem: ideas of the proof

Lifting of the Haar wavelets



Extension theorem: ideas of the proof

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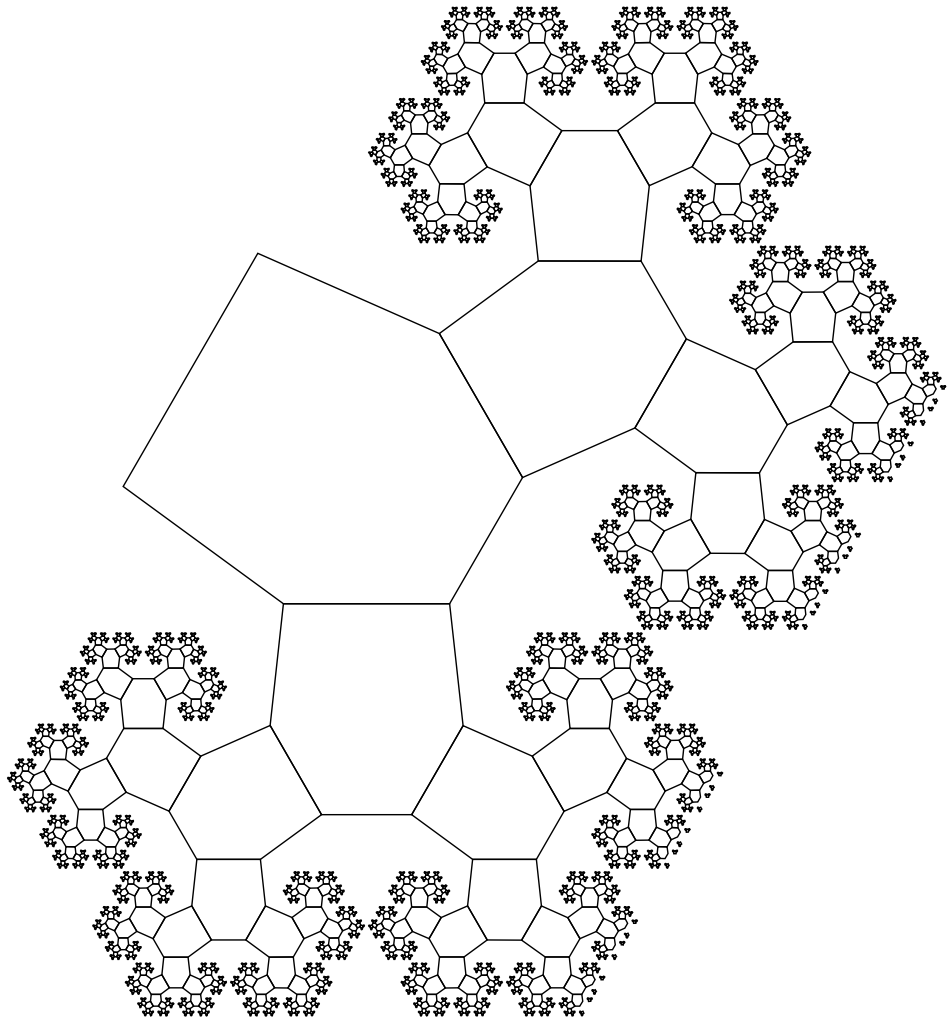

Extension theorem: ideas of the proof

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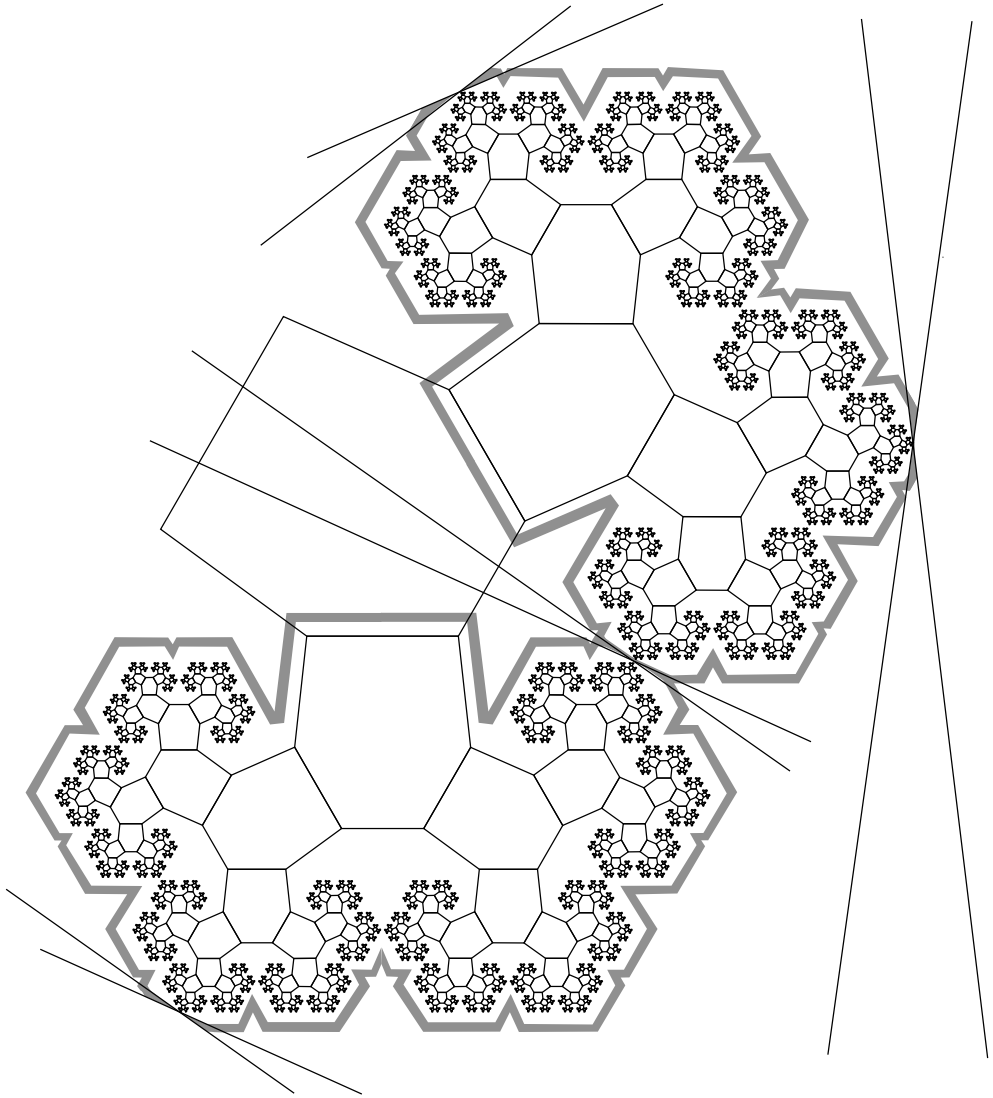

Extension theorem: ideas of the proof

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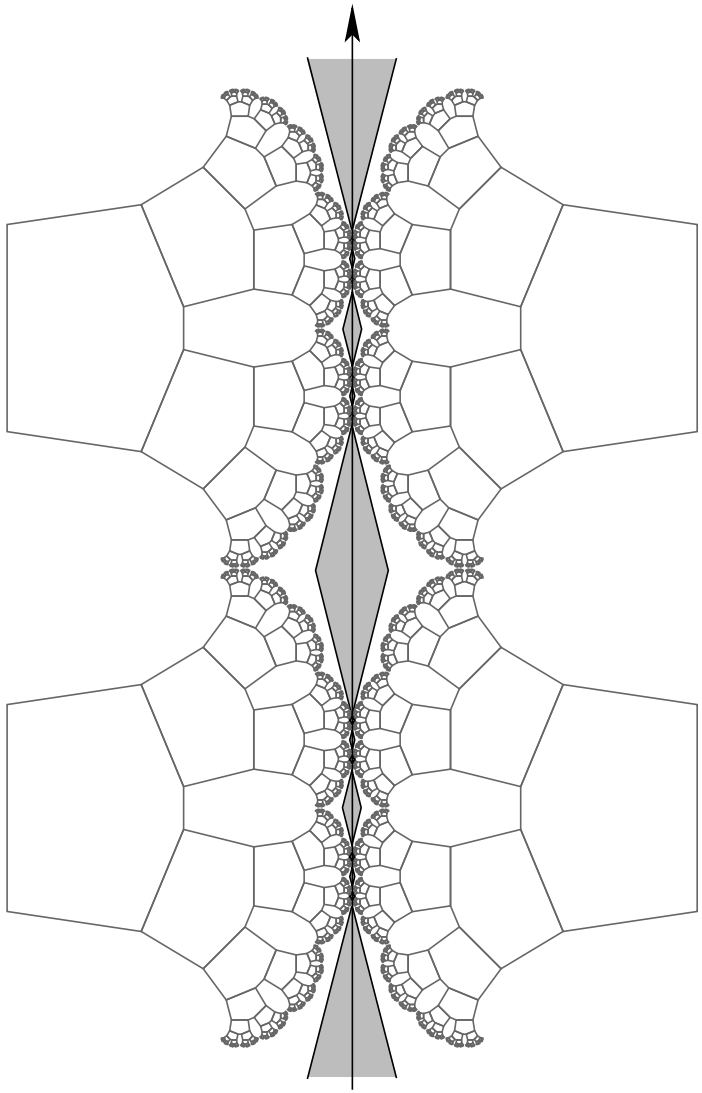
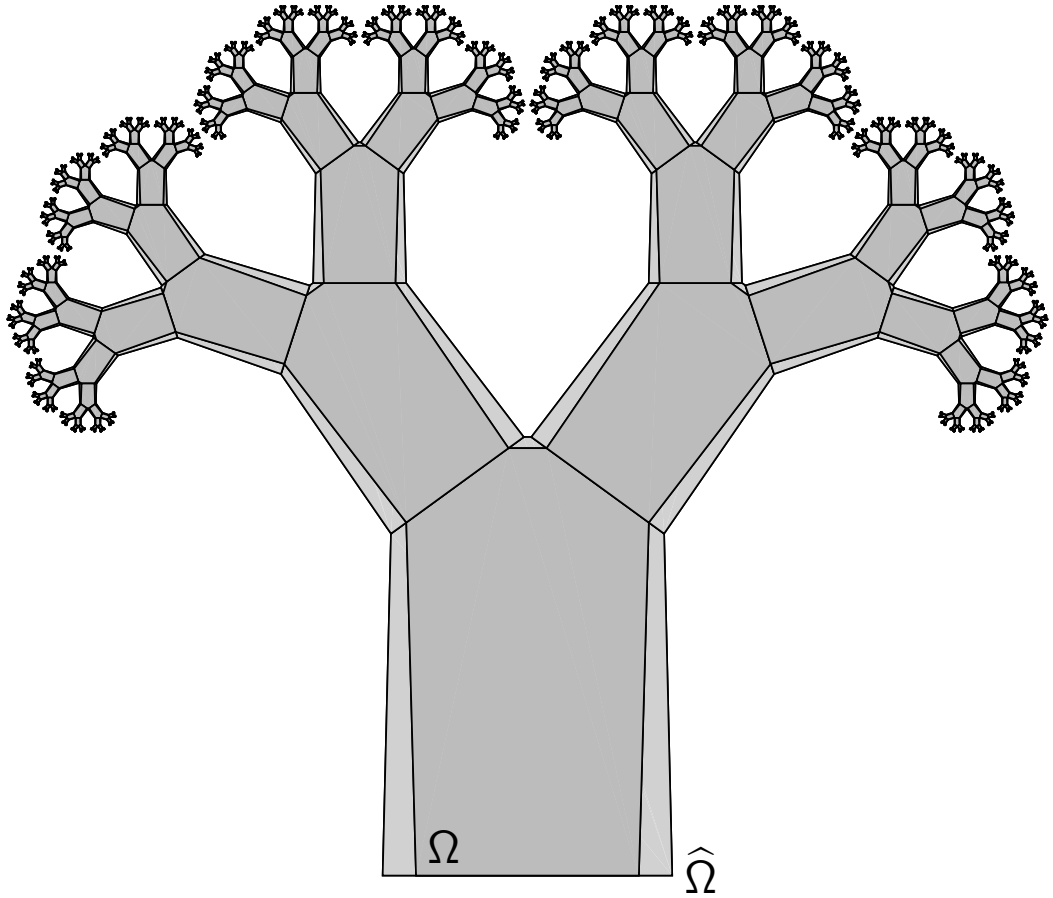


Extension theorem: ideas of the proof

Step 2

Consruct a new ramified domain $\widehat{\Omega}$ containing Ω.



Proposition

There exists a linear continuous extension operator

$$\mathcal{F} : W^{1,p}(\Omega) \to W^{1,p}(\widehat{\Omega}).$$

Extension theorem: ideas of the proof

Step 3

Define a sequence of operators $\Lambda_n : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$ such that

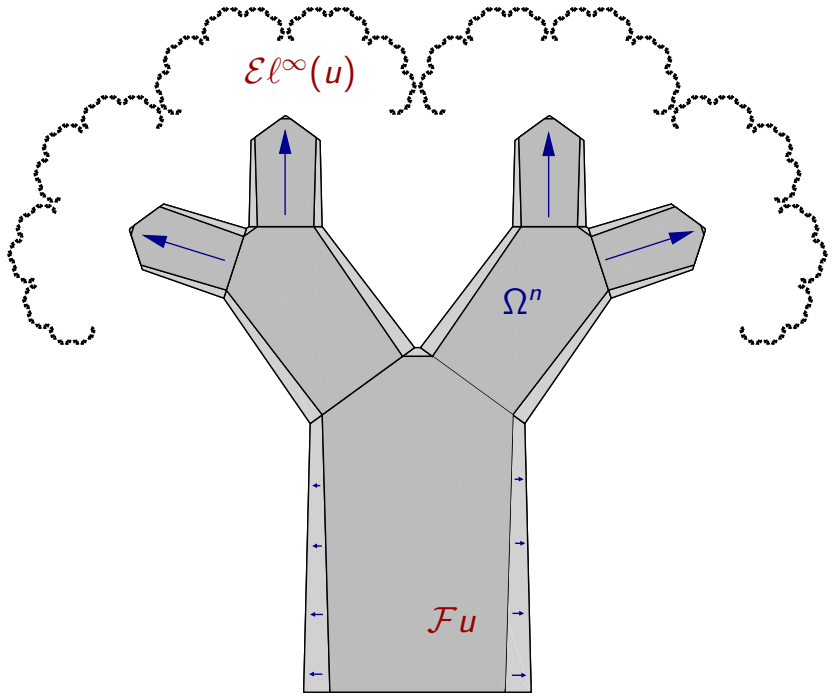
$$(\Lambda_n u)_{|\Omega^n} = u,$$

where Ω^n is the ramified domain truncated after the nth generation.

Λ_n is defined by

$$\Lambda_n u = \rho_n \mathcal{F} u + (1 - \rho_n)\mathcal{E}\ell^\infty(u).$$

where ρ_n is a cut-off function.



Proposition

The sequence of operators Λ_n converges to an extension operator Λ.

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Comparison of the notions of trace

The following theorem justifies *a posteriori* the use of several notions of trace on Γ.

Theorem (Y. Achdou, T.D., N. Tchou, 2013)

For $1 < p < \infty$, every function $u \in W^{1,p}(\Omega)$ is strictly defined μ-almost everywhere on Γ, and

$$\overline{u}_{|\Gamma} \;=\; \ell^{\infty}(u)$$

μ-almost everywhere on Γ.

The proof uses as a key ingredient the extension operator for $p < p^{\star}$. In this case, the trace on Γ does not depend on the extension operator: $(\mathcal{E}u)_{|\Gamma} =: \overline{u}_{|\Gamma}$.

Corollary

If $p > p^{\star}$, Ω is not a $W^{1,p}$-extension domain.

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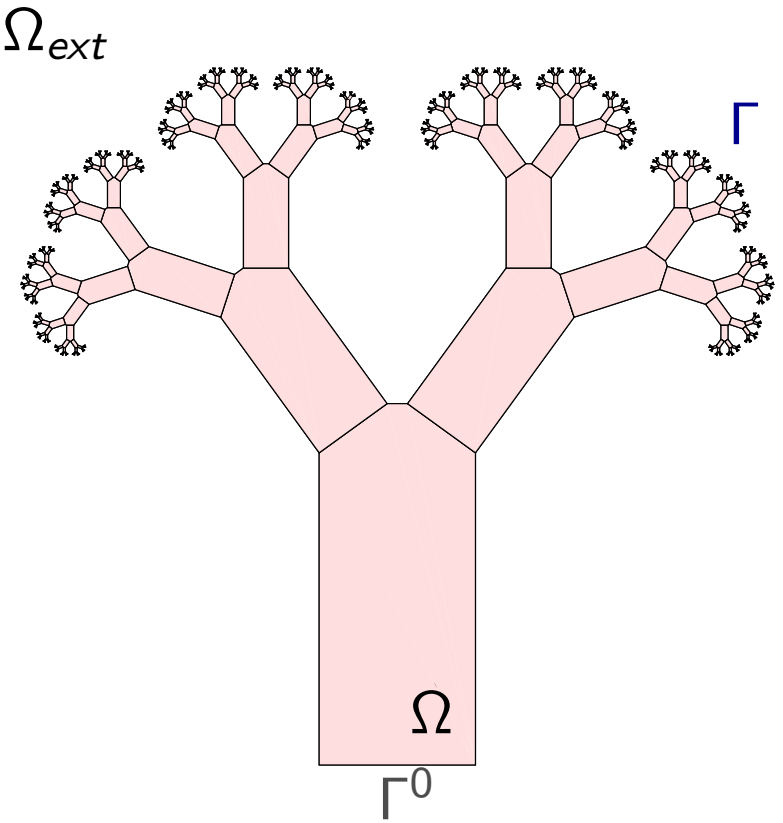
Formal formulation of the problem

$$
\begin{cases}
\Delta u_{int} = f_{int} & \text{in } \Omega \\
\Delta u_{ext} = f_{ext} & \text{in } \Omega_{ext} \\
\partial_n u_{int} = 0 & \text{on } \partial\Omega \backslash \Gamma \\
\partial_n u_{ext} = 0 & \text{on } \partial\Omega \backslash \Gamma \\
\partial_n u_{ext} = 0 & \text{on } \partial B \\
[u] = 0 & \text{on } \Gamma \\
[\partial_n u] = 0 & \text{on } \Gamma
\end{cases}
$$

$$
\begin{aligned}
[u] &= u_{int|\Gamma} - u_{ext|\Gamma} \\
[\partial_n u] &= \partial_n u_{int|\Gamma} - \partial_n u_{ext|\Gamma}
\end{aligned}
$$

$\overline{\Omega} \cup \overline{\Omega}_{ext} = B$ bounded in $\mathbb{R}^2$



Function space:

$$V = \left\{ u \in L^2(B),\ u_{int} \in H^1(\Omega),\ u_{ext} \in H^1(\Omega_{ext}),\ u_{int|\Gamma} = u_{ext|\Gamma} \right\},$$

where we write $u_{int} = u_{|\Omega}$ and $u_{ext} = u_{|\Omega_{ext}}$ for all $u \in L^2(B)$.

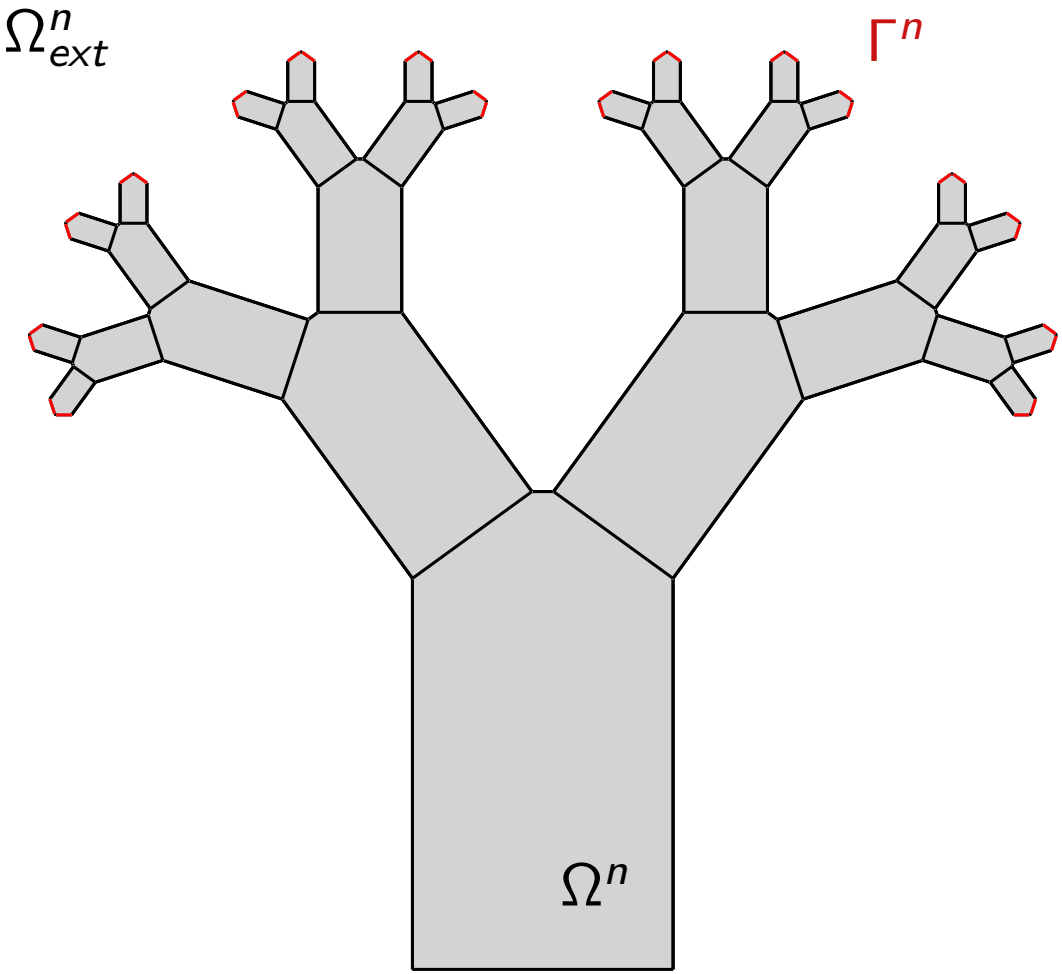
Variational formulation

$$\mathcal{A}(u, v) = \int_\Omega f_{int} v_{int} + \int_{\Omega_{ext}} f_{ext} v_{ext}, \quad \forall v \in V$$

where

$$\mathcal{A}(u, v) = \int_\Omega \nabla u_{int} \cdot \nabla v_{int} + \int_{\Omega_{ext}} \nabla u_{ext} \cdot \nabla v_{ext}$$

The problem with prefractal interface



Truncated domain at generation n

Function space:

$$V^n = \left\{ u \in L^2(B),\ u^n_{int} \in H^1(\Omega^n),\ u_{ext} \in H^1(\Omega^n_{ext}),\ u^n_{int|\Gamma} = u^n_{ext|\Gamma} \right\},$$

where we write $u^n_{int} = u_{|\Omega^n}$ and $u^n_{ext} = u_{|\Omega^n_{ext}}$ for all $u \in L^2(B)$.

Variational formulation

$$\mathcal{A}^n(u,v) \ = \ \int_{\Omega^n} f_{int}\, v^n_{int} + \int_{\Omega^n_{ext}} f_{ext}\, v^n_{ext}, \quad \forall v \in V^n$$

where

$$\mathcal{A}^n(u,v) \ = \ \int_{\Omega^n} \nabla u^n_{int} \cdot \nabla v^n_{int} + \int_{\Omega^n_{ext}} \nabla u^n_{ext} \cdot \nabla v^n_{ext}$$

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Convergence of the energy forms

Definition (Mosco-Convergence)

We say the sequence of forms $(\mathcal{A}_n)$ M-converges to the form $\mathcal{A}$ in $L^2(B)$ if

- *for every sequence (u_n) converging weakly to u in $L^2(B)$,*

$$\underline{\lim}\ \mathcal{A}_n(u_n, u_n) \ \geqslant\ \mathcal{A}(u,u) \quad \text{as } n \to \infty,$$

- *for every function $u \in L^2(B)$, there exists a sequence (u_n) converging strongly in $L^2(B)$ such that*

$$\overline{\lim}\ \mathcal{A}_n(u_n, u_n) \ \leqslant\ \mathcal{A}(u,u) \quad \text{as } n \to \infty.$$

- U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, 1969.

The subcritical case

Theorem (Y. Achdou, T.D., 2014)

In the case $a < a^\star$, the sequence of energy forms $(\mathcal{A}_n)$ converges in the sense of Mosco to the energy form $\mathcal{A}$.

The subcritical case

Theorem (Y. Achdou, T.D., 2014)

In the case $a < a^\star$, the sequence of energy forms $(\mathcal{A}_n)$ converges in the sense of Mosco to the energy form $\mathcal{A}$.

Ideas of the proof:

- Take (u_n) converging weakly to u in $L^2(B)$, we may suppose

$$\mathcal{A}_n(u_n, u_n) \leqslant c$$

for c independent from n. Then, $u_n \in V_n \subset V$.
Up to a subsequence, $u_n \rightharpoonup v$ in V, for some $v \in V$. Then $u = v$, and $\mathcal{A}(u, u) \leqslant \underline{\lim}\ \mathcal{A}_n(u_n, u_n)$.

The subcritical case

Theorem (Y. Achdou, T.D., 2014)

In the case $a < a^\star$, the sequence of energy forms $(\mathcal{A}_n)$ converges in the sense of Mosco to the energy form $\mathcal{A}$.

Ideas of the proof:

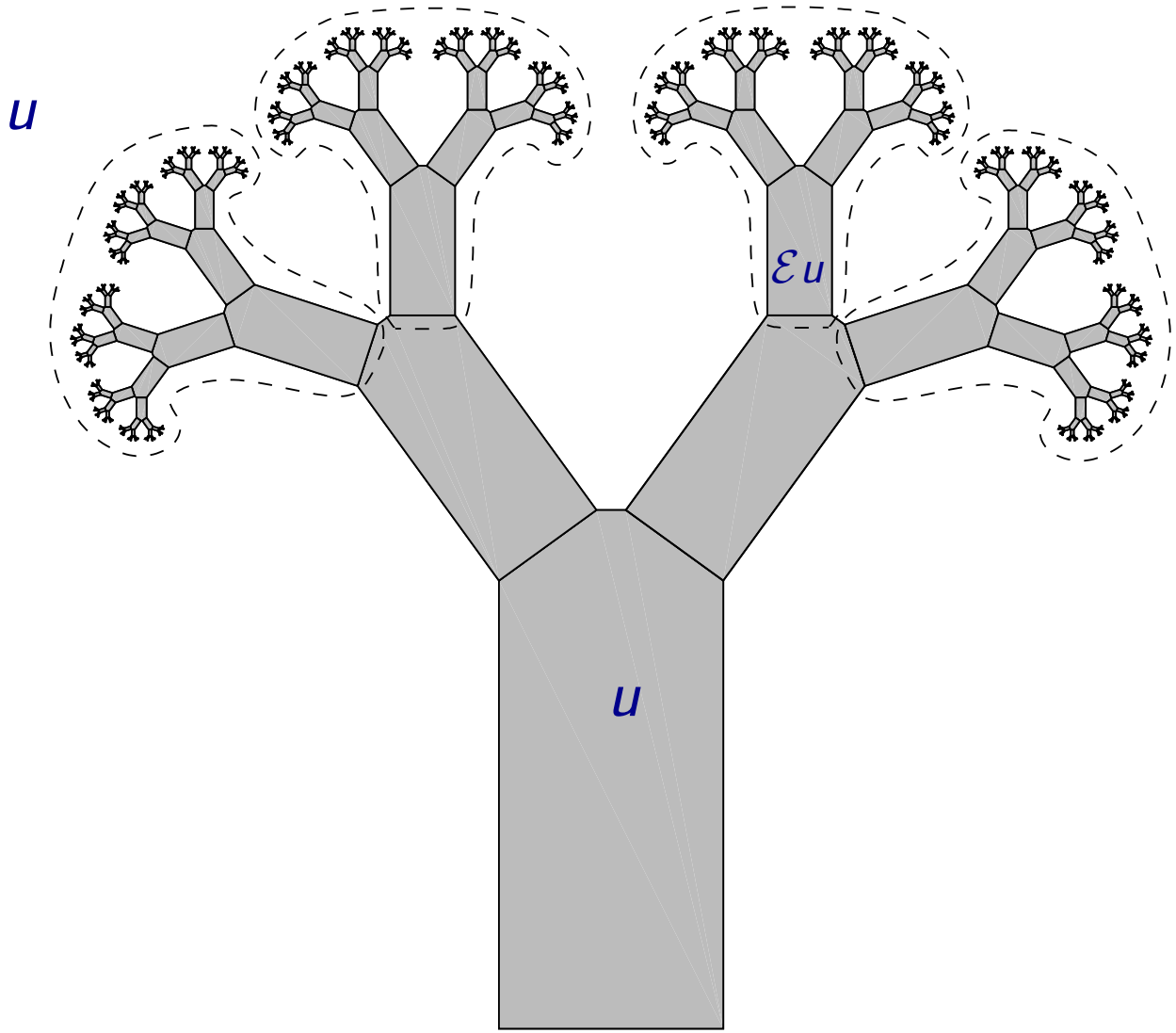
- Take (u_n) converging weakly to u in $L^2(B)$, we may suppose

$$\mathcal{A}_n(u_n, u_n) \leqslant c$$

for c independent from n. Then, $u_n \in V_n \subset V$.
Up to a subsequence, $u_n \rightharpoonup v$ in V, for some $v \in V$. Then $u = v$, and $\mathcal{A}(u,u) \leqslant \underline{\lim}\ \mathcal{A}_n(u_n, u_n)$.

- Take $u \in V$. We use the extension operator $H^1(\Omega) \to H^1(B)$ to construct a sequence (u_n) in V_n such that $u_n \to u$ in V as $n \to \infty$.

The subcritical case



The critical case

Problem: In general, there is no extension operator

$$H^1(\Omega) \ \rightarrow \ H^1(\mathbb{R}^2)$$

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The critical case

Problem: In general, there is no extension operator

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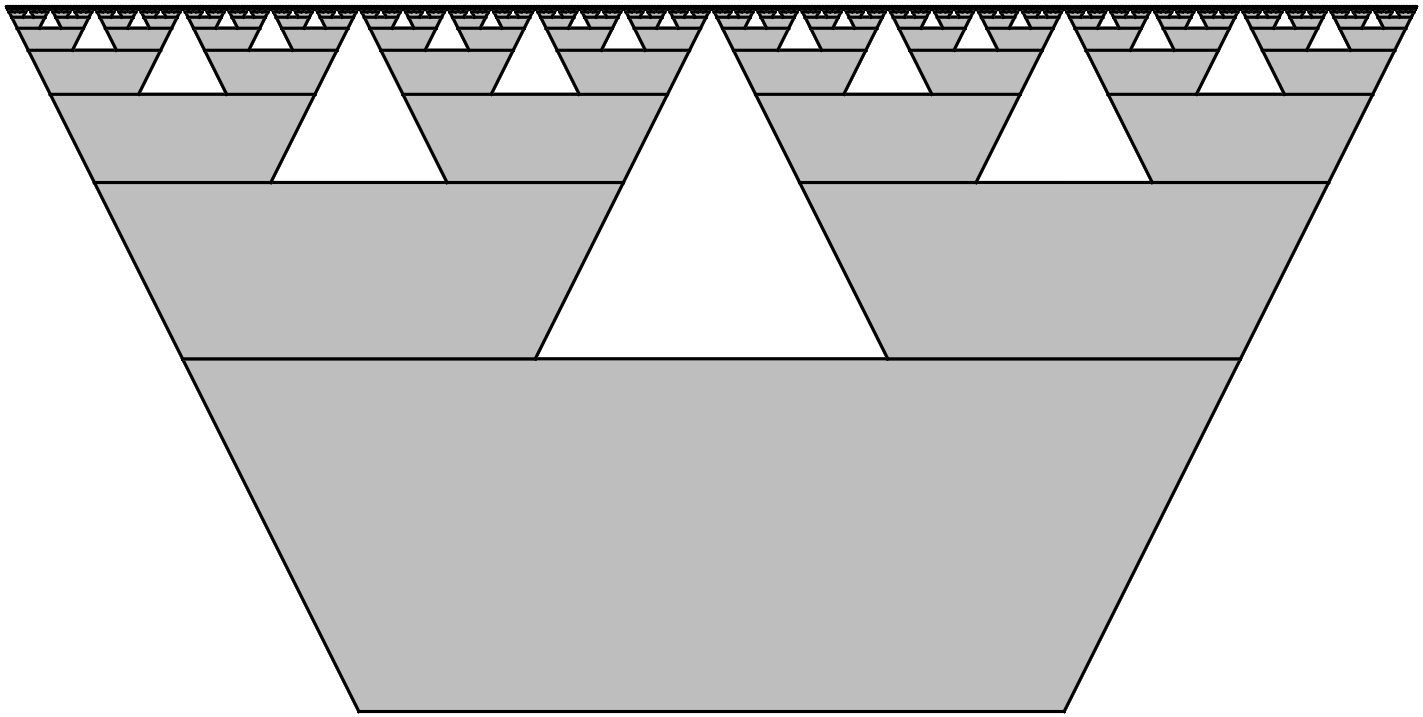
in the critical case $a = a^\star$.

Idea: Use the transmission condition to obtain more regularity:

$$u_{int}|_\Gamma \ = \ \underbrace{u_{ext}|_\Gamma}_{\in H^{1/2}(\Gamma)}$$

in the case $\theta = 0$.

The critical case when $\theta = 0$



The critical case when $\theta = 0$

Theorem (Y. Achdou, T.D., 2014)

If $u \in H^1(\Omega)$, and $u_{|\Gamma} \in H^{1/2}(\Gamma)$, then there exists an extension $\tilde{u} \in H^1(\mathbb{R}^2)$ of u such that

$$\|\tilde{u}\|_{H^1(\mathbb{R}^2)} \leqslant C\|u\|_{H^1(\Omega)},$$

where the constant $C > 0$ is independent from u.

- P. Auscher, N. Badr, *Sobolev spaces on multiple cones*, Annales de la Faculté des Sciences de Toulouse, 2010.

The general case: the transmission condition

Canopy:

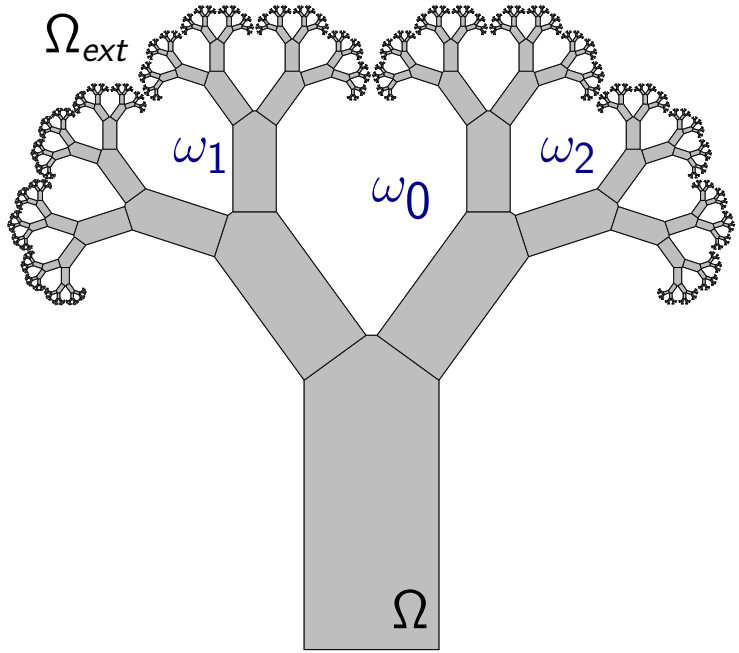
$$\gamma := \Gamma \cap \partial\Omega_{ext}$$

It can be shown that

$$d_c := \dim_H \gamma < \dim_H \Gamma$$

Also introduce:

$$\gamma_\sigma := \Gamma \cap \partial\omega_\sigma$$



$$\omega_\sigma := f_\sigma(\omega_0)$$

for $\sigma \in \bigcup_n \{1,2\}^n$

Transmission condition:

$$\begin{cases} u_{int|\gamma} = u_{ext|\gamma} \\ u_{int|\gamma_\sigma} = u_{ext|\gamma_\sigma} \text{ for all } \sigma \in \bigcup_n \{1,2\}^n \end{cases}$$

Perspectives

- The 3D case
  - trace results, extension?
  - particular geometries?
- Link between trace, extension and the dimension of the multiple points of the set Γ
- Transmission problems
  - Convergence of the energy forms in the general case
  - Other transmission problems

Thank you for your attention...

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... and a very happy birthday to Kenneth!