On the Lipschitz Equivalence of Fractals

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Section 1

Introduction

Lipschitz equivalence ($X \simeq Y$)

$$\text{metric space } X \xrightarrow{f \text{ bijection}} \text{metric space } Y$$

$$C^{-1} \leq \frac{d_Y\big(f(x_1), f(x_2)\big)}{d_X(x_1, x_2)} \leq C$$

$\dim_{\mathrm{H}} X = \dim_{\mathrm{H}} Y$: same size

$X \simeq Y$: same geometric structure

Falconer & Marsh (1992)

Fractal geometry is sometimes thought of as the study of equivalence classes of sets under bi-Lipschitz mappings.

Lipschitz equivalence of fractals

- There is little known about the Lipschitz equivalence of fractals, even for self similar sets in Euclidean spaces.
- Lipschitz equivalence $\Longrightarrow$ equal dimension + topology equivalence. But the inverse is false.

Basic Problem

$$\begin{cases} \text{equal dimension} \\ \text{topology equivalence} \end{cases} + \text{what conditions} \Longrightarrow \text{Lipschitz equivalence}\,?$$

An example: equal dimension + topology equivalence $\not\Rightarrow$ Lipschitz equivalence

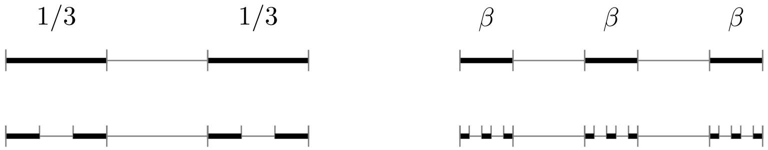
Example (Falconer & Marsh, 1992)

Let $\mathbf{C}$ be the middle-third Cantor set and E the self-similar set satisfying

$$E = \beta E \cup \big(\beta E + (1-\beta)/2\big) \cup \big(\beta E + (1-\beta)\big),$$

where $\beta = 3^{-\log 3/\log 2}$. Then

$$\dim_{\mathrm{H}} E = \dim_{\mathrm{H}} \mathbf{C}, \quad \text{but} \quad E \not\simeq \mathbf{C}.$$



Falconer & Marsh's Theorems

Theorem (Falconer & Marsh, 1989)

Quasi-circles C_1 and C_2 are Lipschitz equivalent if and only if $\dim_{\mathrm{H}} C_1 = \dim_{\mathrm{H}} C_2$.

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Suppose IFS $\mathcal{S}, \mathcal{T}$ both satisfy the SSC and $r_1, \ldots, r_n$ are ratios of $\mathcal{S}$, $t_1, \ldots, t_m$ are ratios of $\mathcal{T}$. If $E_{\mathcal{S}} \simeq E_{\mathcal{T}}$, then

1. $\dim_{\mathrm{H}} E_{\mathcal{S}} = \dim_{\mathrm{H}} E_{\mathcal{T}} = s$;
2. $\mathbb{Q}^* \log r_1 + \cdots + \mathbb{Q}^* \log r_n = \mathbb{Q}^* \log t_1 + \cdots + \mathbb{Q}^* \log t_m$;
3. $\mathbb{Q}(r_1^s, \ldots, r_n^s) = \mathbb{Q}(t_1^s, \ldots, t_m^s)$.

Here $\mathbb{Q}^*$ = nonnegative rational numbers.

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Here $\mathbb{Q}^* =$ *nonnegative rational numbers.*

- Falconer & Marsh's work provided the basic idea to study the problem.
- Their result implied that Lipschitz equivalence of self-similar sets is heavily dependent on the algebraic properties of ratios.

Developments: strong separation condition

Known algebraic properties of ratios determine Lipschitz equivalence.

Unknown exactly what algebraic properties affect Lipschitz equivalence.

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Unknown exactly what algebraic properties affect Lipschitz equivalence.

Necessary condition

Cooper & Pignataro, 1988 measure linear.

Falconer & Marsh, 1992 two necessary conditions, measure linear in general case.

Rao, Ruan & Wang, 2012 another necessary conditions, resolve some special cases.

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Necessary and sufficient condition

Xi, 2010 graph-directed system.

Llorente & Mattila, 2010 Bi-Lipschitz embedding.

Deng, Wen, Xiong & Xi, 2011 Bi-Lipschitz embedding.

But none of above necessary and sufficient conditions are based on the algebraic properties of ratios and so it is difficult to verify them.

Developments: open set condition & totally disconnected

$\{1, 3, 5\}$-$\{1, 4, 5\}$ problem

Algebraic properties & geometric structure affect Lipschitz equivalence.

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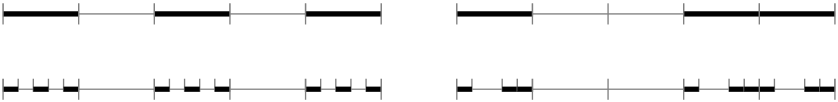

$\{1,3,5\}$-$\{1,4,5\}$ problem, by David & Semmes

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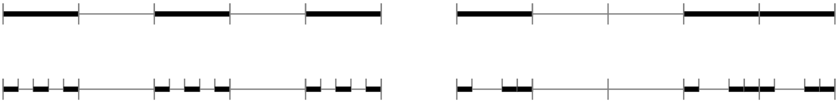

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Different radios

Xi & Ruan, 2007 equivalence if and only if $\log r_1 / \log r_3 \in \mathbb{Q}$.

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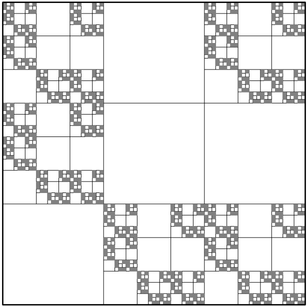
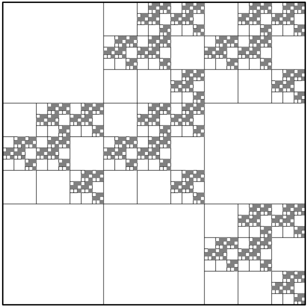
Generated $\{1,3,5\}$-$\{1,4,5\}$ problem





Xi & Xiong, 2010 higher dimensional Euclidean spaces.

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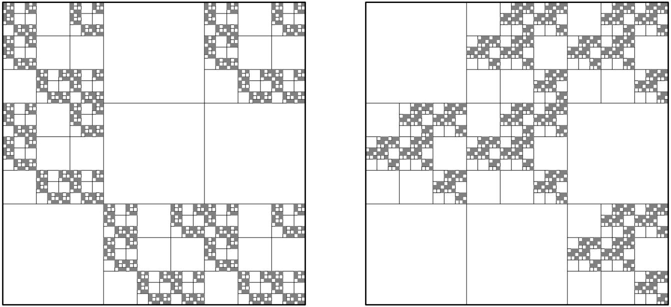

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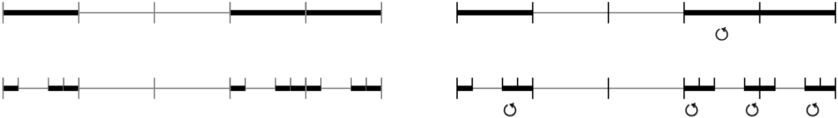

Xiong & Xi, 2012 rotation.

Section 2

Lipschitz equivalence class and ideal class

IFS families $\mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}$ and $\mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}(p, r)$

Consider similar IFS $\mathcal{S}$ satisfying the following four conditions:

1. TDC: the self-similar set is totally disconnected;
2. OSC: the open set condition;
3. defined on the Euclidean spaces.
4. commensurable: the ratios $r_1, \ldots, r_n$ of $\mathcal{S}$ satisfy $\log r_i / \log r_j \in \mathbb{Q}$.

Let $\mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}$ denotes the set of all such IFSs.

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Let $\mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}$ denotes the set of all such IFSs.

For IFS $\mathcal{S} \in \mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}$, let $r_{\mathcal{S}} \in (0, 1)$ determined by

$$\mathbb{Z} \log r_{\mathcal{S}} = \mathbb{Z} \log r_1 + \cdots + \mathbb{Z} \log r_n.$$

Write $p_{\mathcal{S}} = r_{\mathcal{S}}^s$, where s is the dimension of the self-similar set generated by $\mathcal{S}$. Define $\mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}(p, r) = \{\mathcal{S} \in \mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}} \colon p_{\mathcal{S}} = p, r_{\mathcal{S}} = r\}$.

Main discovery

Theorem (Xi & Xiong, arXiv:1304.0103)

Assume $\mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}(p,r) \neq \emptyset$. There is a one-to-one correspondence between the Lipschitz equivalence classes of $\mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}(p,r)$ and the ideal classes of $\mathbb{Z}[p]$.

- This result connects a geometrical object (Lipschitz equivalence classes) with a algebraic object (ideal class).
- This result reveals an interesting relationship between Lipschitz equivalence problem in fractal geometry and the Gauss class number problem in algebraic number theory.

Ideal

- $I \subset R(+, \cdot)$ is a ideal if
  (i) $(I, +)$ is a group; (ii) $a \cdot I \subset I$ for all $a \in R$.
- An ideal $I \subset R$ is called principal if $I = aR$ for some $a \in R$.
- A ring R is called a principal ideal domain if all the ideals are principal.

Example

Every ideal of $\mathbb{Z}$ has the form $n \cdot \mathbb{Z}$ for some $n \in \mathbb{Z}$. And so $\mathbb{Z}$ is a principal ideal domain.

Example

Ideal $(2, \sqrt{10}) \subset \mathbb{Z}[\sqrt{10}]$ is not a principal ideal. And so $\mathbb{Z}[\sqrt{10}]$ is not a principal ideal domain.

Ideal class and class number

I, J: two ideals of R.

- $I \sim J$: if $aI = bJ$ for some $a, b \in R$;
- *ideal classes*: the corresponding equivalence classes;
- *class number*: the cardinal number of ideal classes.

Example

All principal ideals are equivalent. And so

$$R \text{ is a principal ideal domain} \iff R \text{ has class number } 1.$$

Example

The class number of $\mathbb{Z}[\sqrt{10}]$ is 2. In fact, for every ideal I of $\mathbb{Z}[\sqrt{10}]$, either $I \sim (2, \sqrt{10})$ or I is a principal ideal.

The ideal of IFS

- Let IFS $\mathcal{S} \in \mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}$, $E_{\mathcal{S}}$ the self-similar set of $\mathcal{S}$, $s = \dim_{\mathrm{H}} E_{\mathcal{S}}$ and $\mu_{\mathcal{S}} = \mathcal{H}^s|_{E_{\mathcal{S}}} / \mathcal{H}^s(E_{\mathcal{S}})$.

Definition (interior separated set)

A compact set $F \subset E_{\mathcal{S}}$ is called an interior separated set if

- (separated) F and $E_{\mathcal{S}} \setminus F$ are both compact;
- (interior) $\exists$ open set O satisfying the OSC and $F \subset O$.

- We remark that $\mu_{\mathcal{S}}(F) \in \mathbb{Z}[p_{\mathcal{S}}]$ for every interior separated set F.

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Definition (ideal of IFS)

$I_{\mathcal{S}}$, the ideal of IFS $\mathcal{S}$, is defined to be the ideal of $\mathbb{Z}[p_{\mathcal{S}}]$ generated by

$$\{\mu_{\mathcal{S}}(F) \colon F \text{ is an interior separated set of } E_{\mathcal{S}}\}.$$

Examples of ideal of IFS (I)

Example

If $\mathcal{S}$ satisfies the SSC, then $I_{\mathcal{S}} = \mathbb{Z}[p_{\mathcal{S}}]$ is a principal ideal.

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If $\mathcal{S}$ satisfies the SSC, then $I_{\mathcal{S}} = \mathbb{Z}[p_{\mathcal{S}}]$ is a principal ideal.

O a δ-neighborhood of $E_{\mathcal{S}}$ with $\delta \ll 1$

$$\Longrightarrow \; O \text{ satisfies the OSC}$$
$$\Longrightarrow \; E_{\mathcal{S}} \subset O \text{ is an interior separated set}$$
$$\Longrightarrow \; 1 = \mu_{\mathcal{S}}(E_{\mathcal{S}}) \in I_{\mathcal{S}} \; \Longrightarrow \; I_{\mathcal{S}} = \mathbb{Z}[p_{\mathcal{S}}]$$

Examples of ideal of IFS (II)

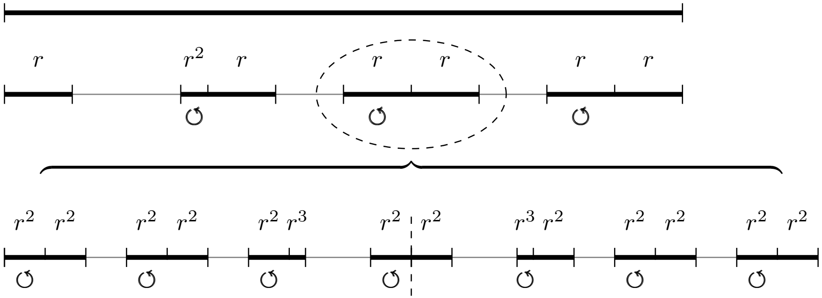
Example

Let $\mathcal{S} = \{S_1, \ldots, S_7\} \in \mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}$, $r = 1/10$ and

$$S_1\colon x \mapsto rx, \quad S_2\colon x \mapsto -r^2x + 3r, \quad S_3\colon x \mapsto rx + 3r,$$
$$S_4\colon x \mapsto -rx + 6r,\ S_5\colon x \mapsto rx + 6r,\ S_6\colon x \mapsto -rx + 9r,\ S_7\colon x \mapsto rx + 9r.$$

Then $p_{\mathcal{S}} = \sqrt{10} - 3$ is a root of $p^2 + 6p = 1$ and

$$I_{\mathcal{S}} = (2, p_{\mathcal{S}} + 1) = (2, \sqrt{10})$$ is not a principal ideal.



Sufficient and necessary condition for Lipschitz equivalence

Theorem (Xi & Xiong, arXiv:1304.0103)

Suppose that $\mathcal{S}, \mathcal{T} \in \mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}$, *then* $E_{\mathcal{S}} \simeq E_{\mathcal{T}}$ *if and only if*

(i) $\dim_{\mathrm{H}} E_{\mathcal{S}} = \dim_{\mathrm{H}} E_{\mathcal{T}}$;

(ii) $\log r_{\mathcal{S}} / \log r_{\mathcal{T}} \in \mathbb{Q}$;

(iii) $I_{\mathcal{S}} = a I_{\mathcal{T}}$ *for some* $a \in \mathbb{R}$.

- In the theorem, the two IFSs $\mathcal{S}$ and $\mathcal{T}$ are allowed to be defined on Euclidean spaces of different dimensions.

  e.g., if $E_{\mathcal{S}} \subset \mathbb{R}^1$ and $E_{\mathcal{T}} \subset \mathbb{R}^2$, we still have

  $$E_{\mathcal{S}} \simeq E_{\mathcal{T}} \iff \text{the above three conditions.}$$

Lipschitz equivalence and ideal equivalence

For $\mathcal{S}, \mathcal{T} \in \mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}(p, r)$, we have

$$\dim_{\mathrm{H}} E_{\mathcal{S}} = \dim_{\mathrm{H}} E_{\mathcal{T}} = \log p / \log r, \quad \mathbb{Z}[p_{\mathcal{S}}] = \mathbb{Z}[p_{\mathcal{T}}] = \mathbb{Z}[p].$$

Theorem (Xi & Xiong, arXiv:1304.0103)

Suppose that $\mathcal{S}, \mathcal{T} \in \mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}(p, r)$, then $E_{\mathcal{S}} \simeq E_{\mathcal{T}}$ if and only if $I_{\mathcal{S}} \sim I_{\mathcal{T}}$.

That means each Lipschitz equivalence class of $\mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}(p, r)$ corresponds to an ideal class of $\mathbb{Z}[p]$.

Corollary (Fact: there are only finitely many ideal classes of $\mathbb{Z}[p]$)

There are only finitely many Lipschitz equivalence classes of $\mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}(p, r)$.

One-to-one correspondence between the two equivalence classes

Theorem (Xi & Xiong, arXiv:1304.0103)

Assume $\mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}(p, r) \neq \emptyset$. *There is a one-to-one correspondence between the Lipschitz equivalence classes of* $\mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}(p, r)$ *and the ideal classes of* $\mathbb{Z}[p]$.

- In fact, for each ideal I of $\mathbb{Z}[p]$, there is an IFS $\mathcal{S} \in \mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}(p, r)$ with $I_{\mathcal{S}} = I$.

One-to-one correspondence between the two equivalence classes

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Question

How many Lipschitz equivalent classes does $\mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}(p,r)$ *contain?*

- This is a geometric version of Gauss class number problems.

Gauss Class number problems

$h(D)$: the class number of the ring of algebraic integers of $\mathbb{Q}(\sqrt{D})$, where D is square-free.

Gauss Class number problems in Disquisitiones Arithmeticae (1801):

- $h(D) \to \infty$ as $D \to -\infty$.

  Proved by Hecke, Deuring, Heilbronn, 1934.

- If $D < 0$ and $h(D) = 1$, then

  $$D \in \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}.$$

  Proved by Heegner, Baker, Stark, 1966.

- There are infinitely many $D > 0$ such that $h(D) = 1$.
  Open.

Lipschitz class number one

$$\mathbb{Z}[p] \text{ is a principal ideal domain} \iff \mathbb{Z}[p] \text{ with class number one}$$
$$\iff \mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}(p, r) \text{ with Lipschitz class number one}$$

$\mathbb{Z}[p]$ is a principal ideal domain when $p = 1/N, \sqrt{2} - 1, (\sqrt{3} - 1)/2, \ldots$.

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$\mathbb{Z}[p]$ is a principal ideal domain when $p = 1/N, \sqrt{2} - 1, (\sqrt{3} - 1)/2, \dots$.

Theorem (Xi & Xiong, arXiv:1304.0103)

Suppose that $\mathcal{S} = \{S_1, \dots, S_N\}$, $\mathcal{T} = \{T_1, \dots, T_N\}$ *and*

- $\mathcal{S}, \mathcal{T}$ *satisfy the OSC;*
- *all the ratios of* S_i *and* T_j *equal to* r*;*
- $E_{\mathcal{S}} \subset \mathbb{R}^d$, $E_{\mathcal{T}} \subset \mathbb{R}^{d'}$ *are totally disconnected.*

Then $E_{\mathcal{S}} \simeq E_{\mathcal{T}}$.

Proof.

$\mathcal{S}, \mathcal{T} \in \mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}}(1/N, r)$ and $\mathbb{Z}[1/N]$ is a principal ideal domain. □

Principal ideal class

- Principal ideal implies simple geometric structure.
- Write $\mathrm{PI} = \{\mathcal{S} \in \mathrm{TDC} \cap \mathrm{OSC}_1^{\mathrm{E}} \colon I_{\mathcal{S}} \text{ is principal}\}$.

Theorem (Xi & Xiong, arXiv:1304.0103)

Suppose that $\mathcal{S}, \mathcal{T} \in \mathrm{PI}$, then $E_{\mathcal{S}} \simeq E_{\mathcal{T}}$ if and only if

(i) $\dim_{\mathrm{H}} E_{\mathcal{S}} = \dim_{\mathrm{H}} E_{\mathcal{T}}$;

(ii) $\log r_{\mathcal{S}} / \log r_{\mathcal{T}} \in \mathbb{Q}$;

(iii) $\mathbb{Z}[p_{\mathcal{S}}] = \mathbb{Z}[p_{\mathcal{T}}]$.

Go to main theorem

Proof.

$$\begin{aligned} I_{\mathcal{S}} = aI_{\mathcal{T}} &\implies \mathbb{Z}[p_{\mathcal{S}}] = b\mathbb{Z}[p_{\mathcal{T}}] \implies b \in \mathbb{Z}[p_{\mathcal{S}}] \\ &\implies b\mathbb{Z}[p_{\mathcal{S}}] \subset \mathbb{Z}[p_{\mathcal{S}}] = b\mathbb{Z}[p_{\mathcal{T}}] \implies \mathbb{Z}[p_{\mathcal{S}}] \subset \mathbb{Z}[p_{\mathcal{T}}]. \end{aligned}$$

□

SSC corresponds to the principal ideal class

Theorem (Xi & Xiong, arXiv:1304.0103)

Suppose that $\mathcal{S}, \mathcal{T}$ both satisfy the SSC and the ratios of them are both commensurable. Then $E_{\mathcal{S}} \simeq E_{\mathcal{T}}$ if and only if

1. $\dim_{\mathrm{H}} E_{\mathcal{S}} = \dim_{\mathrm{H}} E_{\mathcal{T}}$;
2. $\log r_{\mathcal{S}} / \log r_{\mathcal{T}} \in \mathbb{Q}$;
3. $\mathbb{Z}[p_{\mathcal{S}}] = \mathbb{Z}[p_{\mathcal{T}}]$. (Ring condition)

Go to theorem of PI

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3. $\mathbb{Z}[p_{\mathcal{S}}] = \mathbb{Z}[p_{\mathcal{T}}]$. (Ring condition)

Go to theorem of PI

Impressive development after Falconer and Marsh's work

Necessary conditions in non-commensurable case (Falconer & Marsh)

Suppose that $\mathcal{S}, \mathcal{T}$ both satisfy the SSC and $r_1, \ldots, r_n$ are ratios of $\mathcal{S}$, $t_1, \ldots, t_m$ are ratios of $\mathcal{T}$. If $E_{\mathcal{S}} \simeq E_{\mathcal{T}}$, then

1. $\dim_{\mathrm{H}} E_{\mathcal{S}} = \dim_{\mathrm{H}} E_{\mathcal{T}} = s$;
2. $\mathbb{Q}^* \log r_1 + \cdots + \mathbb{Q}^* \log r_n = \mathbb{Q}^* \log t_1 + \cdots + \mathbb{Q}^* \log t_m$;
3. $\mathbb{Q}(r_1^s, \ldots, r_n^s) = \mathbb{Q}(t_1^s, \ldots, t_m^s)$. (Field condition)

Example: the ring condition stronger than the field condition

Example (assume $\mathcal{S}, \mathcal{T}$ satisfy the SSC)

- Ratios of $\mathcal{S}$: 3^{-1}, 3^{-1}, 3^{-2} and 3^{-2};
- $p_{\mathcal{S}} = (\sqrt{3}-1)/2$: the positive solution of $2p_{\mathcal{S}}^2 + 2p_{\mathcal{S}} = 1$;
- Ratios of $\mathcal{T}$: $\underbrace{3^{-3}, \dots, 3^{-3}}_{20}, \underbrace{3^{-6}, \dots, 3^{-6}}_{8}$.
- $p_{\mathcal{T}} = (3\sqrt{3}-5)/4$: the positive solution of $8p_{\mathcal{T}}^2 + 20p_{\mathcal{T}} = 1$;

Then $\log p_{\mathcal{S}} / \log p_{\mathcal{T}} = \frac{1}{3} \in \mathbb{Q}$,

$\mathbb{Q}(p_{\mathcal{S}}) = \mathbb{Q}(p_{\mathcal{T}}) = \mathbb{Q}(\sqrt{3})$, but $\mathbb{Z}[p_{\mathcal{S}}] = \mathbb{Z}[\sqrt{3}, \frac{1}{2}] \neq \mathbb{Z}[p_{\mathcal{T}}] = \mathbb{Z}[3\sqrt{3}, \frac{1}{2}]$.

SSC without commensurable condition

Example

Ratios of E_1 $1/9$ and $4/9$

Ratios of E_2 $1/81, 1/81, 1/81, 1/81$ and $4/9$

$\ldots$ $\ldots$

Ratios of E_n $\underbrace{9^{-n}, \ldots, 9^{-n}}_{3^{n-1}}$ and $4/9$

$\ldots$ $\ldots$

1. $\dim_{\mathrm{H}} E_n = 1/2$;
2. $\mathbb{Q}^* \log 9^{-n} + \mathbb{Q}^* \log(4/9) = \mathbb{Q}^* \log 9 + \mathbb{Q}^* \log(4/9)$;
3. $\mathbb{Z}[3^{-n}, 2/3] = \mathbb{Z}[1/3]$;

But we can show that $E_m \not\simeq E_n$ for $m \neq n$.

SSC without commensurable condition

Example

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Ratios of E_2 $1/81, 1/81, 1/81, 1/81$ and $4/9$

$\dots$ $\dots$

Ratios of E_n $\underbrace{9^{-n}, \dots, 9^{-n}}_{3^{n-1}}$ and $4/9$

$\dots$ $\dots$

1. $\dim_{\mathrm{H}} E_n = 1/2$;
2. $\mathbb{Q}^* \log 9^{-n} + \mathbb{Q}^* \log(4/9) = \mathbb{Q}^* \log 9 + \mathbb{Q}^* \log(4/9)$;
3. $\mathbb{Z}[3^{-n}, 2/3] = \mathbb{Z}[1/3]$;

But we can show that $E_m \not\simeq E_n$ for $m \neq n$.

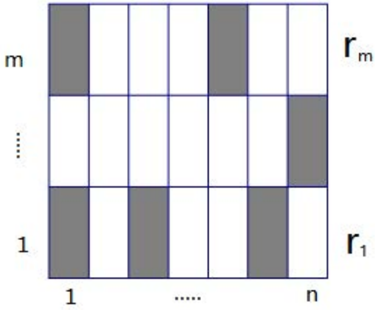
Question

What is the sufficient and necessary condition for the Lipschitz equivalence for self-similar sets satisfying the SSC?

Section 3

Lipschitz equivalence of self-affine sets

McMullen Bedford Carpet



r: Number of chosen rectangles

$$\dim_H E = \log \sum_{j=0}^{m-1} r_j^{\log m/\log n} / \log m.$$

We use $\mathcal{R}(n, m, r, r_0, \cdots, r_{m-1})$ to denote the collection of all such McMullen sets, i.e., with the number of rectangles in each line fixed.

Dust-like set

Let $E \in \mathcal{R}(n, m, r, r_0, \cdots, r_{m-1})$. We call the McMullen set E *dust-like*, if $S_i(E) \cap S_j(E) = \emptyset$ for all $i \neq j \in \{0, \cdots, r-1\}$. We denote by $\mathcal{DR}(n, m, r, r_0, \cdots, r_{m-1})$ the collection of all dust-like McMullen sets in

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Theorem (Li, Li& Miao)

Let E and F be two McMullen sets in $\mathcal{DR}(n, m, r, r_0, \cdots, r_{m-1})$. Then the sets E and F are Lipschitz equivalent.

HBSC set

We say that the McMullen set E satisfies *horizontal block separation condition (HBSC)* if one of the following properties holds:
(I) Let $J \neq J' \in \{S_i(Q) : i = 0, \cdots, r-1\}$. Then J and J' are disjoint;
(II) Let $J \neq J' \in \{S_i(Q), S_i(Q) + (1,0) : i = 0, \cdots, r-1\}$ such that $J \cap J' \neq \emptyset$. Then J and J' lie in the same horizontal line.
We write $\mathcal{SR}(n, m, r, r_0, \cdots, r_{m-1})$ for the collection of all McMullen sets in $\mathcal{R}(n, m, r, r_0, \cdots, r_{m-1})$ satisfying HBSC.

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