Falconer-Sloan condition and random affine code tree fractals

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Introduction: singular value function

- Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a non-singular contracting linear map with singular values

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- Define the singular value function by

$$\Phi^s(T) = \begin{cases} \sigma_1\sigma_2\cdots\sigma_{m-1}\sigma_m^{s-m+1}, & \text{if } 0 \leq s \leq d, \\ \sigma_1\sigma_2\cdots\sigma_{d-1}\sigma_d^{s-d+1}, & \text{if } s > d. \end{cases}$$

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- Φ^s is submultiplicative, i.e. $\Phi^s(TS) \leq \Phi^s(T)\Phi^s(S)$

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- There are different approaches to overcoming problems caused by this, for example, the invariant cone condition (Feng and Shmerkin), irreducibility (Feng), non-existence of parallelly mapped vectors and a general condition introduced by Falconer and Sloan.

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Theorem (Falconer 1988)

Assume that $\|T_i\| < 1/3$ for all i. Then for $\mathcal{L}^{Md}$-almost all $\mathbf{a} \in \mathbb{R}^{Md}$

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- Solomyak: 1/3 can be replaced by 1/2.
- Przytycki and Urbański: 1/2 is the best possible bound.

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- In both cases there is total independence both is space, i.e. between different nodes at a fixed construction level, and in scale or time, i.e. once a node is chosen its descendants are chosen independently of the previous history.
- Random affine code tree fractals have certain independence only in time direction.

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Falconer-Sloan condition guarantees that this does not happen simultaneously for all maps in the family.

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Define the inner product $\langle \cdot \mid \cdot \rangle$ on Λ^m by the formula

$$\langle \mathbf{v} \mid \mathbf{w} \rangle \omega = \mathbf{v} \wedge *\mathbf{w}.$$

Here ω is the normalised volume form on $\mathbb{R}^d$ and $* : \Lambda^m \to \Lambda^{d-m}$ is the Hodge star operator.

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Any linear map $S : \mathbb{R}^d \to \mathbb{R}^d$ induces a linear map $S : \Lambda^m \to \Lambda^m$ such that $S(v_1 \wedge \cdots \wedge v_m) = Sv_1 \wedge \cdots \wedge Sv_m$ for all $v_1 \wedge \cdots \wedge v_m \in \Lambda_0^m$.

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Definition

Let $m \in \mathbb{N}$ with $0 \leq m \leq d$. The family $\{S_i\}_{i \in I}$ satisfies condition $C(m)$ if for all $\mathbf{v}, \mathbf{w} \in \Lambda_0^m \setminus \{0\}$ there is $i \in I$ such that $\langle S_i \mathbf{v} \mid \mathbf{w} \rangle \neq 0$.

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Let $0 < s < d$ be non-integral and let m be the integer part of s. The family $\{S_i\}_{i \in I}$ satisfies condition $C(s)$ if for all $\mathbf{v}, \mathbf{w} \in \Lambda_0^m \setminus \{0\}$ and $\mathbf{v} \wedge v, \mathbf{w} \wedge w \in \Lambda_0^{m+1} \setminus \{0\}$ there is $i \in I$ such that $\langle S_i \mathbf{v} \mid \mathbf{w} \rangle \neq 0$ and $\langle S_i(\mathbf{v} \wedge v) \mid \mathbf{w} \wedge w \rangle \neq 0$.

Remark

(1) The family $\{S_i\}_{i\in I}$ satisfies condition $C(m)$ if and only if for all $\mathbf{v} \in \Lambda_0^m \setminus \{0\}$ the set $\{S_i\mathbf{v} \mid i \in I\}$ spans Λ^m.

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Goal: to prove that for generic pairs (F, G) of linear maps the family of compositions of F and G up to a certain level satisfies $C(s)$ for all $0 \leq s \leq d$.

Genericity of Falconer-Sloan condition

- Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map with d different real eigenvalues $\{\lambda_1, \ldots, \lambda_d\}$ such that for all $k = 1, \ldots, d$

  $$\lambda_{i_1} \cdots \lambda_{i_k} \neq \lambda_{j_1} \cdots \lambda_{j_k}$$

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- Let A be an invertible $d \times d$-matrix and let $\{\tilde{e}_1, \dots, \tilde{e}_d\}$ be a basis of $\mathbb{R}^d$ such that $\tilde{e}_i = Ae_i$ for all $1 \leq i \leq d$.

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Theorem (Li, Stenflo, J^2)

The family $\mathcal{S}_{2n_0^2}$ satisfies $C(m)$ and $C(s)$ for all $m = 1, \ldots, d$ and $0 < s < d$ provided that $A \in \mathcal{M}_d$.

We identify the space of families $\mathcal{F} = \{S_i : \mathbb{R}^d \to \mathbb{R}^d\}_{i=1}^k$ of linear maps with $\mathbb{R}^{d^2 k}$ and define

$$\mathcal{S}_l(\mathcal{F}) = \{S_{i_1} \circ \cdots \circ S_{i_j} \mid 1 \leq j \leq l \text{ and } S_{i_m} \in \mathcal{F} \text{ for all } 1 \leq m \leq j\}.$$

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Corollary (Li, Stenflo, J[2])

The set

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Note that the upper bound for the number of iterates needed to satisfy the condition $C(s)$ is independent of the original family $\mathcal{F}$.

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- Examples: attractors of graph directed Markov systems generated by affine maps, or more generally, sub-self-affine sets.

Theorem (Koivusalo, Käenmäki, Stenflo, Suomala, J^2)

Assuming that $\|T_i^\lambda\| \leq \sigma < \frac{1}{2}$ for all $\lambda \in \Lambda$ and $i = 1, \ldots, M_\lambda$, we have for all ω

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This result can be generalised (and the proof can be simplified) by replacing (3) with a probabilistic version of the Falconer-Sloan condition. In particular, (1) and (4) are not needed.

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- The upper bound $\frac{1}{2}$ is optimal.
- When $d = 2$ assumption (**3**) implies (**3′**).
- In general (**3′**) is weaker than (**3**).

Happy birthday, Kenneth!