Falconer-Sloan condition and random affine code tree fractals

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Introduction: singular value function

• Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a non-singular contracting linear map with singular values

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$$\Phi^{s}(T) = \begin{cases} \sigma_{1}\sigma_{2}\cdots\sigma_{m-1}\sigma_{m}^{s-m+1}, & \text{if } 0 \leq s \leq d, \\ \sigma_{1}\sigma_{2}\cdots\sigma_{d-1}\sigma_{d}^{s-d+1}, & \text{if } s > d. \end{cases}$$

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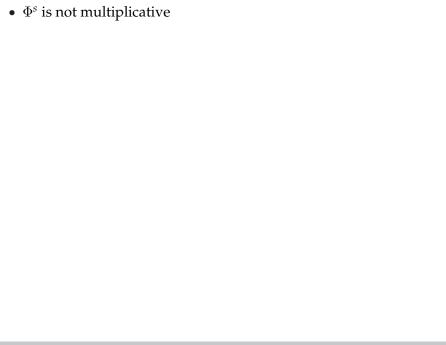
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• Φ^s is submultiplicative, i.e. $\Phi^s(TS) \leq \Phi^s(T)\Phi^s(S)$



- Φ^s is not multiplicative
- There are different approaches to overcoming problems caused by this, for example, the invariant cone condition (Feng and Shmerkin), irreducibility (Feng), non-existence of parallelly mapped vectors and a general condition introduced by Falconer and Sloan.

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$$p(s) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{(\mathbf{i}_1, \dots, \mathbf{i}_n) \in \{1, \dots, M\}^n} \Phi^s(T_{\mathbf{i}_1} \circ \dots \circ T_{\mathbf{i}_n}).$$

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Theorem (Falconer 1988)

Assume that $||T_i|| < 1/3$ for all i. Then for \mathcal{L}^{Md} -almost all $\mathbf{a} \in \mathbb{R}^{Md}$

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- Przytycki and Urbański: 1/2 is the best possible bound.

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- In both cases there is total independence both is space, i.e. between different nodes at a fixed construction level, and in scale or time, i.e. once a node is chosen its descendants are chosen independently of the previous history.
- Random affine code tree fractals have certain independence only in time direction.

Falconer-Sloan condition

Let *A* and *B* be $d \times d$ -matrices. Let $v \in \mathbb{R}^d$ such that |v| = 1 and ||B|| = |Bv|.

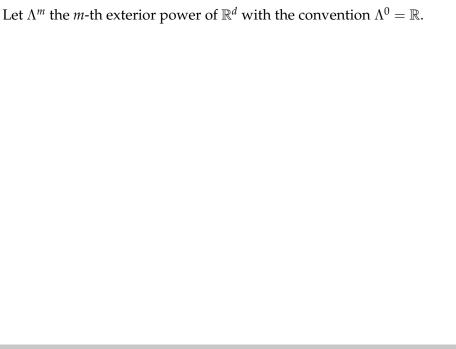
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Falconer-Sloan condition guarantees that this does not happen simultaneously for all maps in the family.



Let Λ^m the m-th exterior power of \mathbb{R}^d with the convention $\Lambda^0 = \mathbb{R}$. Let Λ^m_0 be the set of decomposable m-vectors, i.e.

$$\Lambda_0^m = \{ \mathbf{v} = v_1 \wedge \cdots \wedge v_m \mid v_i \in \mathbb{R}^d \}.$$

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Define the inner product $\langle \cdot | \cdot \rangle$ on Λ^m by the formula

$$\langle \mathbf{v} \mid \mathbf{w} \rangle \omega = \mathbf{v} \wedge *\mathbf{w}.$$

Here ω is the normalised volume form on \mathbb{R}^d and $*: \Lambda^m \to \Lambda^{d-m}$ is the Hodge star operator.

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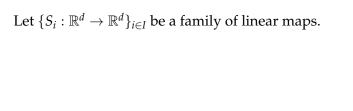
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Any linear map $S : \mathbb{R}^d \to \mathbb{R}^d$ induces a linear map $S : \Lambda^m \to \Lambda^m$ such that $S(v_1 \wedge \cdots \wedge v_m) = Sv_1 \wedge \cdots \wedge Sv_m$ for all $v_1 \wedge \cdots \wedge v_m \in \Lambda_0^m$.



Let ${S_i : \mathbb{R}^d \to \mathbb{R}^d}_{i \in I}$ be a family of linear maps.

Definition

Let $m \in \mathbb{N}$ with $0 \le m \le d$. The family $\{S_i\}_{i \in I}$ satisfies condition C(m) if for all $\mathbf{v}, \mathbf{w} \in \Lambda_0^m \setminus \{0\}$ there is $i \in I$ such that $\langle S_i \mathbf{v} \mid \mathbf{w} \rangle \ne 0$.

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Let 0 < s < d be non-integral and let m be the integer part of s. The family $\{S_i\}_{i \in I}$ satisfies condition C(s) if for all $\mathbf{v}, \mathbf{w} \in \Lambda_0^m \setminus \{0\}$ and $\mathbf{v} \wedge v, \mathbf{w} \wedge w \in \Lambda_0^{m+1} \setminus \{0\}$ there is $i \in I$ such that $\langle S_i \mathbf{v} \mid \mathbf{w} \rangle \neq 0$ and $\langle S_i(\mathbf{v} \wedge v) \mid \mathbf{w} \wedge w \rangle \neq 0$.

Remark

(1) The family $\{S_i\}_{i\in I}$ satisfies condition C(m) if and only if for all $\mathbf{v} \in \Lambda_0^m \setminus \{0\}$ the set $\{S_i\mathbf{v} \mid i \in I\}$ spans Λ^m .

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Goal: to prove that for generic pairs (F, G) of linear maps the family of compositions of F and G up to a certain level satisfies C(s) for all $0 \le s \le d$.

• Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map with d different real eigenvalues $\{\lambda_1, \dots, \lambda_d\}$ such that for all $k = 1, \dots, d$

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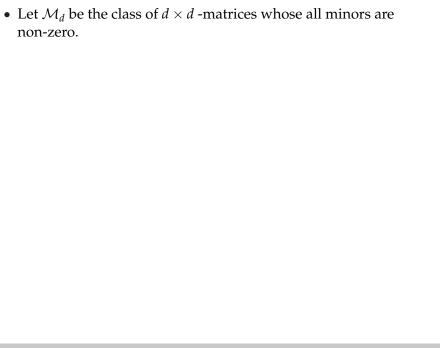
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The family $S_{2n_0^2}$ satisfies C(m) and C(s) for all m = 1, ..., d and 0 < s < d provided that $A \in \mathcal{M}_d$.

We identify the space of families $\mathcal{F} = \{S_i : \mathbb{R}^d \to \mathbb{R}^d\}_{i=1}^k$ of linear maps with \mathbb{R}^{d^2k} and define

$$S_l(\mathcal{F}) = \{S_{i_1} \circ \cdots \circ S_{i_j} \mid 1 \leq j \leq l \text{ and } S_{i_m} \in \mathcal{F} \text{ for all } 1 \leq m \leq j\}.$$

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Corollary (Li, Stenflo, J²)

The set

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Note that the upper bound for the number of iterates needed to satisfy the condition C(s) is independent of the original family \mathcal{F} .

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- Examples: attractors of graph directed Markov systems generated by affine maps, or more generally, sub-self-affine sets.

Assuming that $||T_i^{\lambda}|| \le \sigma < \frac{1}{2}$ for all $\lambda \in \Lambda$ and $i = 1, \dots, M_{\lambda}$, we have for all ω

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- Define $d^{\omega} = \inf\{s \mid \mathcal{M}^s(\Sigma^{\omega}) = 0\} = \sup\{s \mid \mathcal{M}^s(\Sigma^{\omega}) = \infty\}.$

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Then P-almost surely the pressure $p^{\tilde{\omega}}$ exists. Moreover, there exists a unique s_0 with $p^{\tilde{\omega}}(s_0) = 0$ P-almost surely.

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This result can be generalised (and the proof can be simplified) by replacing (3) with a probabilistic version of the Falconer-Sloan condition. In particular, (1) and (4) are not needed.

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- When d = 2 assumption (3) implies (3').
- In general (3') is weaker than (3).

Happy birthday, Kenneth!