Multistable Lévy motions and their continuous approximations

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Outline

My presentation includes 4 parts:

1. Functional central limit theorem for multistable Lévy motions
2. Stochastic Hölder continuity and strong localisability
3. Continuous approximation of MsLM
4. Integrals of multistable Lévy measure

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Definition of Lévy Motions

Definition. A stochastic process $\{L(t), t \geq 0\}$ is called (standard) $\alpha-$stable *Lévy motion* if the following three conditions hold:

1) $L(0) = 0$ a.s.;

2) L has independent increments;

3) $L(t) - L(s) \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0)$ for any $0 \leq s < t < \infty$ and for some $0 < \alpha \leq 2, -1 \leq \beta \leq 1$, where $S_\alpha(\sigma, \beta, 0)$ stands for a stable random variable with index of stability α, scale parameter σ and skewness parameter β.

3′) L has stationary increments;

3″) For any $\varepsilon > 0$ and $t \geq 0$ it holds

$$\lim_{h \to 0} \mathbb{P}(|L(t+h) - L(t)| \geq \varepsilon) = 0.$$

Why introduce Multistable Lévy Motions?

Since $\alpha-$stable Lévy motions have stationary increments and constant stability index α, they cannot be used for describing some real-world phenomena, such as financial records, internet traffic, noise on telephone lines and atmospheric noise. In these real-world phenomena, the local regularity and the local stability levels may vary with time. Thus it is natural to set up a class of processes such that their local regularity and local stability level vary with a parameter t. Such processes are useful both theoretically and in practice.

An example of a process having varying stability index, called multistable processes, was introduced in Falconer and Lévy Véhel (2009).

One says that $\{X(t), t \in \mathbb{R}\}$ is $h-$localisable at $u \in \mathbb{R}, h > 0$, if there exists a non-trivial process X'_u such that

$$\lim_{r \searrow 0} \frac{X(u+rt) - X(u)}{r^h} = X'_u(t),$$

where convergence is in finite dimensional distributions.

Definition. *A stochastic process* $\{X(t), t \in \mathbb{R}\}$ *is called multistable if for almost all u, X is localisable at u with* X'_u *an* $\alpha-$*stable process for some* $\alpha = \alpha(u)$, *where* $0 < \alpha(u) \leq 2$.

Let $D[0,1]$ be the set of "càdlàg" functions on $[0,1]$ endowed with the Skorohod metric d_S; see Billingsley (1968).

One says that X is $h-$strongly localisable at $u, h > 0$, with strong local form X'_u, if X and X'_u have versions in $D[0,1]$ and the convergence

$$\lim_{r \searrow 0} \frac{X(u+rt) - X(u)}{r^h} = X'_u(t)$$

is in distribution with respect to d_S; see Falconer and Lévy Véhel (2009).

Constructions of multistable processes

After the work of Falconer and Lévy Véhel, some constructions of multistable processes have been established.

Poisson representation:

Falconer and Lévy Véhel (2009)

Ferguson-Klass-LePage series representation:

Le Guével and Lévy Véhel (2012,2013)

Poisson + FKL series representations:

Le Guével, Lévy Véhel and Liu (2013)

Multistable measure defined by characteristic function:

Falconer and Liu (2012)

Certain properties of multistable processes are investigated in those papers.

Falconer and Liu (2012) proved that, for all $(\theta_1, ..., \theta_d) \in \mathbb{R}^d$,

$$\mathbb{E} \exp \left\{ i \Big(\sum_{j=1}^{d} \theta_j \int f_j(x) M(dx) \Big) \right\} = \exp \left\{ - \int \Big| \sum_{j=1}^{d} \theta_j f_j(x) \Big|^{\alpha(x)} dx \right\} \quad (1.1)$$

defines a consistent probability distribution on the functions f_j of

$$\mathcal{F}_{a,b} = \left\{ f : \int_{-\infty}^{\infty} |f(x)|^a dx, \int_{-\infty}^{\infty} |f(x)|^b dx < \infty \right\}.$$

Moreover, the integral $\int f(x) M(dx)$ is well-defined, and the integrals of functions with disjoint supports are independent. One calls M the multistable Lévy measure.

In particular, if $\Big(\alpha(x)-\alpha(x+t)\Big)\ln t \to 0$ uniformly for all x in finite interval as $t \searrow 0$, then the integral $L_{\alpha(u)}(u)=\int \mathbf{1}_{[0,u]}(x)M(dx)$ defines a symmetric multistable Lévy motion (MsLM) on the positive half-line.

The aim of this presentation:

We give a functional central limit theorem for symmetric MsLM. Our theorem is based on weighted sums of independent random variables, which is different from the previous constructions. Then we show that MsLM are stochastic Hölder continuous and strongly localisable. Moreover, a continuous approximation of MsLM and a new representation for the integrals of multistable Lévy measure are established.

We give a functional central limit theorem for symmetric MsLM.

Theorem 1

Let $\alpha(u) \in D[0,1]$ *ranging in* $[a,b] \subset (0,2]$. *Assume that* $(X(\frac{k}{2^n}))_{n\in\mathbb{N},\ k=1,...,2^n}$ *is a family of independent random variables with* $X(\frac{k}{2^n}) \sim S_{\alpha(\frac{k}{2^n})}(1,0,0)$. *Then*

$$L_{\alpha(u)}(u) = \lim_{n\to\infty} \sum_{k=1}^{\lfloor 2^n u \rfloor} \left(\frac{1}{2^n}\right)^{1/\alpha(\frac{k}{2^n})} X\left(\frac{k}{2^n}\right), \qquad u \in [0,1], \tag{1.2}$$

converges in distribution with respect to d_S. *Its joint characteristic function is given as follows: for all* $\theta_j \in \mathbb{R}$ *and* $u_j \in [0,1]$, $j = 1, ..., d$,

$$\mathbb{E} \exp\left\{ i \sum_{j=1}^{d} \theta_j L_{\alpha(u_j)}(u_j) \right\} = \exp\left\{ -\int \left| \sum_{j=1}^{d} \theta_j \mathbf{1}_{[0,u_j]}(s) \right|^{\alpha(s)} ds \right\}. \tag{1.3}$$

Remark 1

Notice that the summands of (1.2) verify:

$$\left(\frac{1}{2^n}\right)^{1/\alpha(\frac{\lfloor 2^n u\rfloor}{2^n})} X\left(\frac{\lfloor 2^n u\rfloor}{2^n}\right) \sim S_{\alpha(\frac{\lfloor 2^n u\rfloor}{2^n})}\left(\left(\frac{1}{2^n}\right)^{1/\alpha(\frac{\lfloor 2^n u\rfloor}{2^n})}, 0, 0\right). \qquad (1.4)$$

Since

$$\lim_{n\to\infty} \alpha\left(\frac{\lfloor 2^n u\rfloor}{2^n}\right) = \alpha(u),$$

equality (1.2) means that the increment at the point u of the process $L_{\alpha(u)}(u)$ behaves locally like an $\alpha(u)-$stable random variable, but with the stability index $\alpha(u)$ varying with u.

If the function α satisfies

$$\Big(\alpha(u) - \alpha(u+t)\Big) \ln t \to 0, \quad \text{as} \;\; t \searrow 0, \tag{1.5}$$

uniformly for all $u \in [0,1]$, then

Theorem 2 (Falconer and Liu (2012))

Assume condition (1.5). Then $L_{\alpha(u)}(u)$ is an $\alpha(u)-$multistable Lévy motion, called independent increments multistable Lévy motion.

This means that the process $L_{\alpha(u)}(u)$ is localisable at u to Lévy's process $L_{\alpha(u)}(t)$ with stability index $\alpha(u)$, i.e.

$$\lim_{r \searrow 0} \frac{L_{\alpha(u+rt)}(u+rt) - L_{\alpha(u)}(u)}{r^{1/\alpha(u)}} = L_{\alpha(u)}(t)$$

in finite dimensional distributions.

Equivalent definition of symmetric $\alpha-$stable Lévy motions $L_\alpha(u)$:

Corollary 3

There is a sequence of i.i.d. symmetric $\alpha-$stable random variables $(Y_k)_{k\in\mathbb{N}}$ with an unit scale parameter such that

$$L_\alpha(u) = \lim_{n\to\infty} \sum_{k=1}^{\lfloor nu \rfloor} \frac{1}{n^{1/\alpha}} Y_k \ , \qquad u \in [0,1], \tag{1.6}$$

converges in distribution with respect to Skorohod metric d_S.

How to construct a MsLM on the whole line?

Let $\alpha(x),\ x \in \mathbb{R}$, be a continuous function ranging in $[a, b] \subset (0, 2]$. Set $\alpha_k(x) = \alpha(x + k),\ x \in [0, 1]$, for all $k \geq 0$. For every $\alpha_k(x)$, by Theorem 2, we construct a MsLM:

$$L_{\alpha_k(x)}(x) : [0, 1] \to \mathbb{R}.$$

Taking a sequence of independent processes $L_{\alpha_k(x)}(x)$, we can define $\{L_{\alpha(x)}(x) : x \geq 0\}$ by gluing together the parts, more precisely by

$$L_{\alpha(x)}(x) = L_{\alpha_{\lfloor x \rfloor}(x - \lfloor x \rfloor)}(x - \lfloor x \rfloor) + \sum_{k=0}^{\lfloor x \rfloor - 1} L_{\alpha_k(1)}(1), \quad \text{for all } x \geq 0.$$

Similarly, for $x < 0$, we can define $L_{\alpha(x)}(x) = L_{\alpha(x)}(-x)$, since $\beta(x) = \alpha(-x)$ is defined on $[0, +\infty)$.

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Definition. *A random process* $X(t)$, $t \in [0,\ 1]$, *is called stochastic Hölder continuous of exponent* $\beta \in (0, 1]$, *if it holds*

$$\lim_{u \to t} \mathbb{P}(|X(u) - X(t)| \geq C\,|u - t|^{\beta}) = 0$$

for all $t \in [0,\ 1]$ *and a positive constant* C.

It is obvious that if $X(t)$ is stochastic Hölder continuous of exponent $\beta_1 \in (0, 1]$, then $X(t)$ is stochastic Hölder continuous of exponent $\beta_2 \in (0, \beta_1]$.

The following theorem shows that MsLM are stochastic Hölder continuous.

Theorem 4

Let $L_{\alpha(u)}(u)$ be defined by Theorem 1. Then for all $u, r \in [0,1], u \neq r$, it holds

$$\mathbb{P}(|L_{\alpha(r)}(r) - L_{\alpha(u)}(u)| \geq |r-u|^{\beta}) \leq C_{a,b}\, |r-u|^{1-\beta b}, \tag{2.7}$$

where $C_{a,b}$ is a constant depending only on a and b, which implies that $L_{\alpha(u)}(u)$ is stochastic Hölder continuous of exponent

$$\beta \in \left(0, \min\left\{1, \frac{1}{b}\right\}\right).$$

The following theorem shows that MsLM are not only localisable but also strongly localisable.

Theorem 5

Assume the condition of Theorem 2. Then $L_{\alpha(u)}(u)$ is $1/\alpha(x)-$strongly localisable at x with strong local form $L_{\alpha(x)}(u)$, the $\alpha(x)-$stable Lévy motion.

This means

$$\lim_{r \searrow 0} \frac{L_{\alpha(x+rt)}(x+rt) - L_{\alpha(x)}(x)}{r^{1/\alpha(x)}} = L_{\alpha(x)}(t)$$

in distribution with respect to Skorohod metric d_S.

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Recall the definition of the “triangle” function:

$$\varphi(t) = \begin{cases} 2t & \text{for } t \in [0, 1/2) \\ 2 - 2t & \text{for } t \in [1/2, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Define $\varphi_{jk}(t) = \varphi(2^j t - k)$, for $j = 0, 1, ...$ and $k = 0, ..., 2^j - 1$, the dilations and translations of $\varphi(t)$.

We first establish a continuous stable processes starting at 0.

Theorem 6

Assume the i.i.d. random variables $(Z_{jk})_{j,k}$ follow a symmetric $\alpha-$stable law with the unit scale parameter. Then, for all d $> 1/\alpha$, the process

$$X(t) = \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} 2^{-jd} Z_{jk} \varphi_{jk}(t), \quad t \in [0,1],$$

is a continuous and symmetric α-stable process.

The scale parameter $\sigma(t)$ of the process $X(t)$ has the following estimation

$$\varphi^{1/\alpha}(t) \;\leq\; \sigma(t) \;\leq\; \left(\frac{1}{1-2^{-\alpha d}}\right)^{1/\alpha}, \qquad t \in [0,1].$$

It is worth noting that when $t = 0, 1$, we have $\sigma(t) = 0$; while when $t \neq 0, 1$, we have $\sigma(t) > 0$. This observation will be useful to establish a continuous approximation of MsLM.

The main idea to construct continuous approximation of MsLM is to replace the stable random variables of Theorem 1 by the continuous stable processes starting at 0.

We give a continuous approximation of MsLM.

Theorem 7

Assume that $\left(X_{\alpha(\frac{k}{2^n})}(t)\right)_{n\in\mathbb{N},k=0,\ldots,2^n-1}$ *is a family of independent and continuous* $\alpha(\frac{k}{2^n})-$*stable random processes. Assume* $X_{\alpha(\frac{k}{2^n})}(0)=0$ *and* $\sigma_{\alpha(\frac{k}{2^n})}(t)>0$ *for all* $t\in(0,1)$ *and all* $n\in\mathbb{N}$, $k=0,\ldots,2^n-1$, *where* $\sigma_{\alpha(\frac{k}{2^n})}(t)$ *is the scale parameter of* $X_{\alpha(\frac{k}{2^n})}(t)$. *Define*

$$\begin{aligned} S_n(u) &= \left(\frac{1}{2^n}\right)^{\alpha(\frac{\lfloor 2^n u\rfloor}{2^n})}\frac{1}{\sigma_{\alpha(\frac{\lfloor 2^n u\rfloor}{2^n})}(\frac{1}{2^n})}X_{\alpha(\frac{\lfloor 2^n u\rfloor}{2^n})}\left(u-\frac{\lfloor 2^n u\rfloor}{2^n}\right) \\ &\quad+\sum_{k=0}^{\lfloor 2^n u\rfloor-1}\left(\frac{1}{2^n}\right)^{\alpha(\frac{k}{2^n})}\frac{1}{\sigma_{\alpha(\frac{k}{2^n})}(\frac{1}{2^n})}X_{\alpha(\frac{k}{2^n})}\left(\frac{1}{2^n}\right), \qquad u\in[0,1]. \end{aligned}$$

Then $(S_n)_{n\in\mathbb{N}}$ *is a continuous approximation of MsLM, that is* $\lim_{n\to\infty}S_n(u)=L_{\alpha(u)}(u)$ *in distribution on* $D[0,1]$.

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Integrals of multistable Lévy measure

Denote by

$$\mathcal{L}_{\alpha(x)}[0,1] = \left\{ f : ||f||_{\alpha(x)} := \inf\left\{ \lambda > 0, \int_0^1 |f(x)/\lambda|^{\alpha(x)} dx = 1 \right\} < \infty \right\}.$$

Note that $||\cdot||_{\alpha(x)}$ is a quasinorm. The following theorem gives a new representation of integrals with respect to multistable Lévy measure.

Theorem 8

Assume that $\alpha(u)$ and $\left(X(\frac{k}{2^n})\right)_{n\in\mathbb{N},\ k=1,\dots,2^n}$ are defined by Theorem 1. Then, for any positive $f \in \mathcal{L}_{\alpha(x)}[0,1]$, it holds

$$\int_0^1 f(x)M(dx) = \lim_{n\to\infty} \sum_{k=1}^{2^n} \left(\frac{1}{2^n}\right)^{1/\alpha(\frac{k}{2^n})} f\left(\frac{k}{2^n}\right) X\left(\frac{k}{2^n}\right) \quad (4.8)$$

in distribution, where M is a multistable measure (Falconer and Liu [6]).

The last theorem relates the convergence of a sequence of multistable integrals to the convergence of the sequence of integrands.

Theorem 9

Let $X_j = \int_0^1 f_j(x)M(dx), j = 1, 2, ...,$ *and* $X = \int_0^1 f(x)M(dx)$*. Then*

$$\lim_{j\to\infty} X_j = X$$

in distribution, if and only if

$$\lim_{j\to\infty} ||f_j(x) - f(x)||_{\alpha(x)} = 0.$$

This theorem shows that convergence of multistable integrals coincides with convergence in quasinorm.

Some work under construction

- Construct MsLM with stochastic stability levels

(For Gaussian processes, see Ayache and Taqqu (2005) for Multifractional Brownian Motion with random exponent.)

- Construct self-stabilizing process

(It means the stability levels of the process are varying with the process itself.)

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Happy birthday Kenneth!