Ph.D. Thesis on
Multistable and Self-stabilizing Processes
and their application in financial modelling

1 Context

Multistable processes are processes which are, at each time t, "tangent" to a stable process with exponent α, where $\alpha = \alpha(t)$ varies with t. These processes have a rich local structure, since both the pointwise Hölder exponent and the local intensity of their jumps may be prescribed independently at each point. Tempered versions of these processes have also been developed, which possess the additional property of having moments of all orders. This makes such processes good candidates for modelling financial records.

The simplest multistable processes are symmetric Lévy multistable motions. Two versions exist: the first one is obtained from a field of Lévy stable processes [3]. More precisely, let $\alpha : [0,1] \to [c,d]$ be a C^2 function, where $[c,d] \subset (1,2)$, $(\Gamma_i)_{i\geq 1}$ be a sequence of arrivals of a Poisson process with intensity one, $(V_i)_{i\geq 1}$ be a sequence of i.i.d. random variables uniformly distributed on $[0,1]$, and $(\gamma_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. The sequences $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, and $(\gamma_i)_{i\geq 1}$ are taken independent. The symmetric Lévy stable field is defined on $[0,1]^2$ as follows [3] :

$$X(u,v) = C_{\alpha(v)}^{1/\alpha(v)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(v)} \mathbf{1}_{[0,u]}(V_i),$$

where

$$C_u = \left(\int_0^\infty x^{-u} \sin(x) dx \right)^{-1}. \tag{1.1}$$

The Lévy multistable motion associated to this field is by definition:

$$Y(t) = X(t,t).$$

Both the above field and process admit a Poisson type representation [2]: let Π be a Poisson process on $[0,1] \times \mathbf{R}$, with mean measure $\mathcal{L}[0,1] \times \mathcal{L}(\mathbf{R})$, where $\mathcal{L}$ is Lebesgue measure. We denote (X,Y) a random point of Π in $[0,1] \times \mathbf{R}$. A representation of the field, which is equivalent in law to the one above, is:

$$X(u,v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} \mathbf{1}_{[0,u]}(\mathsf{X}) \mathsf{Y}^{<-1/\alpha(v)>}$$

where $a^{<b>} = \text{sign}(a)|a|^b$, and where the sum above means:

$$X(u,v) = \lim_{n\to\infty} \sum_{(\mathsf{X},\mathsf{Y})\in\Pi\cap R_n} \mathbf{1}_{[0,u]}(\mathsf{X})\mathsf{Y}^{<-1/\alpha(v)>},$$

when $\alpha(v) \geq 1$, where R_n is the rectangle $\{(x,y) : 0 \leq x \leq 1, |y| \leq n\} \subset \mathbf{R}^2$.

Lévy multistable process is tangent, at each t, to usual Lévy stable motion with index $\alpha(t)$, denoted $L_{\alpha(t)}$, in the following sense [2, 3] :

$$\lim_{r\to 0} \frac{Y(u+rt) - Y(u)}{r^h} = L_{\alpha(t)}(t),$$

where the limit in the sense of finite dimensional distributions.

Furthermore, at each t, the pointwise Hölder exponent of Y is almost surely equal to $\frac{1}{\alpha(t)}$ [4].

A different multistable Lévy motion, that we shall denote Z, may be defined from its joint characteristic function [1]:

$$\mathbb{E}\left(\exp\left(i\sum_{j=1}^{d}\theta_j Z(t_j)\right)\right) = \exp\left(-\int\left|\sum_{j=1}^{d} a(t_j)\theta_j 1_{[0,t_j]}(s)\right|^{\alpha(s)} ds\right).$$

In contradistinction to Y, Z is an independent increments process. It is also tangent at each point t to a Lévy stable motion with exponent $\alpha(t)$. Y et Z are both semi-martingales [5].

Tempered versions of both Y and Z, which generalize the celebrated CGMY process used in financial modelling, are obtained by modifying in an adequate way the definitions above [6].

2 Ph.D. Thesis topic

Numerical tests on financial data show that tempered multistable motions provide sensible models. By adjusting the α function, one is able to reproduce various regimes characterized by different local jump intensities.

For such models to be used in practice, e.g. for option pricing or risk management, a further step is however needed: indeed, they require knowing the future evolution of $\alpha(t)$. Investigating this matter will be the topic of the Ph.D. thesis. Two paths will be explored:

1. <u>Autonomous evolution of α</u>

   In the first variant, we will mimic stochastic volatility models, with the difference that the evolution of α, rather than the one of volatility, will be modelled by a stochastic differential equation (SDE). Thus, we will write a system of coupled SDEs, one for the log-price, and a second one for α. Evolution equations of Hull and White or Heston types will be studied. Existence and uniqueness results will be sought for, and the behaviour of the solutions will be studied, in view of obtaining pricing rules and VaR estimations.

2. <u>Self-stabilizing processes</u>

   An intriguing feature appears when one analyses certain records such as the S&P500 index: it seems that there exits a relation between the value of the index and the one of the local intensity of jumps, as measured by α. This calls for the development of self-stabilizing models, *i.e.* processes X verifying a functional equation of the form: $\alpha(t) = g(X(t))$, almost surely for all t, where g is a smooth deterministic function. All the information concerning the future evolution of α is then incorporated in g, which may be estimated from historical data under the assumption that the relation between X and α does not vary in time.

   This class of models is in a sense analogous to local volatility models: instead of having the local volatility depend on X, it is the local intensity of jumps that does so.

   Self-stabilizing processes may be obtained in three ways: the first one is a fixed point approach starting from a symmetric Lévy stable field. The second one uses a wavelet construction. Finally, the third one builds the process as a solution of a particular SDE.

   As above, the work will focus on the properties of the processes constructed in either of these three ways, and studying pricing and risk management under such models.

3 Skills and Profile

A strong background in stochastic analysis as well as in financial mathematics is required. Moderate skills in programming would be a plus.

4 Contact

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5 Place of work

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References

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