Multifractal and higher dimensional zeta functions

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Abstract

In this paper, we generalize the zeta function for a fractal string (as in [18]) in several directions.

We first modify the zeta function to be associated with a sequence of covers instead of the usual definition involving gap lengths. This modified zeta function allows us to define both a multifractal zeta function and a zeta function for higher-dimensional fractal sets. In the multifractal case, the critical exponents of the zeta function $\zeta(q, s)$ yield the usual multifractal spectrum of the measure. The presence of complex poles for $\zeta(q, s)$ indicate oscillations in the continuous partition function of the measure, and thus give more refined information about the multifractal spectrum of a measure. In the case of a self-similar set in $\mathbb{R}^n$, the modified zeta function yields asymptotic information about both the "box" counting function of the set and the n-dimensional volume of the ϵ-dilation of the set.

1 Background and Motivations

The theory of fractal strings has been developed over the past years by M. Lapidus and co-workers in a series of papers, including [6, 10, 12, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 32]. See also the book [18].

A fractal string $\mathcal{L}$ is simply a bounded open subset Ω of $\mathbb{R}$. Ω may be written as a disjoint union of intervals $I_j = (a_j, b_j)$, *i.e.* $\Omega = \bigcup_{j=1}^{\infty} I_j$. Often, no distinction is made between the open set and its sequence of lengths. By abuse of notation, we will thus speak of the fractal string as $\mathcal{L} = \{\ell_j\}_{j=1}^{\infty}$, where ℓ_j are the lengths of the I_j. We will also denote by $\{l_n\}_{n=1}^{\infty}$ the distinct lengths of $\mathcal{L}$ with multiplicities m_n. Thus, a possible definition of a fractal string is:

Definition 1 *A fractal string $\mathcal{L}$ is an at most countable, non-increasing sequence of lengths whose sum is finite.*

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To a fractal string is associated a complex function, the (geometric) zeta function of $\mathcal{L}$:

Definition 2 *The geometric zeta function of a fractal string $\mathcal{L}$ is*

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} m_n l_n^s$$

for values $s \in \mathbb{C}$ such that the series converges.

The interest of $\zeta_{\mathcal{L}}$ is that it provides rich information about the fractal structure of $\mathcal{L}$, as is developed in the papers mentioned above. The most basic fact is that the critical exponent for $\zeta_{\mathcal{L}}$ is related to the Minkowski or box dimension of the boundary of Ω. Recall the following definition of the Minkowski dimension:

The one-sided volume of the tubular neighborhood of radius ε of $\partial\Omega$ is

$$V(\varepsilon) = \mathrm{vol}_1\{x \in \Omega | \ \ d(x, \partial\Omega) < \varepsilon\}.$$

The *Minkowski dimension* of $\partial\Omega$ is

$$D = D_{\mathcal{L}} := \inf\{\alpha \geq 0 | \ \ V(\varepsilon) = O(\varepsilon^{\alpha-1}) \ \text{ as } \ \varepsilon \to 0^+\}.$$

Notice that the Minkowski dimension is translation invariant: As a consequence, only the lengths of Ω (that is, the associated fractal string $\mathcal{L}$) affect its value. It is thus relevant to speak of the Minkowski dimension of $\mathcal{L}$.

If $\lim_{\varepsilon\to 0^+} V(\varepsilon)\varepsilon^{\alpha-1}$ exists for some α, $\mathcal{L}$ is said to be *Minkowski measurable*, in which case

$$D = \inf\{\alpha \geq 0 | \ \ \lim_{\varepsilon\to 0^+} V(\varepsilon)\varepsilon^{1-\alpha} < \infty\}.$$

The *Minkowski content* of $\mathcal{L}$ is then defined as $\mathcal{M}(D, \mathcal{L}) = \lim_{\varepsilon\to 0^+} V(\varepsilon)\varepsilon^{D-1}$.

The following theorem describes the relation between the Minkowski dimension of a fractal string $\mathcal{L}$ and the sum of each of its lengths with exponent $\sigma \in \mathbb{R}$.

Theorem 1 *([18], Theorem 1.10)*

$$D = \inf\{\sigma \in \mathbb{R} | \ \ \sum_{j=1}^{\infty} \ell_j^{\sigma} < \infty\}.$$

This sum is monotonically decreasing, so there is a unique such $D \in \mathbb{R}$. We see that D is the abscissa of convergence of $\zeta_{\mathcal{L}}(s)$, where $s \in \mathbb{C}$. In fact, D is the largest real pole of $\zeta_{\mathcal{L}}(s)$.

A standing assumption in all that follows is that the zeta function $\zeta_{\mathcal{L}}$ has a meromorphic extension to an open region of the complex plane which extends strictly to the left of line $Re(z) = D$. By an abuse of notation, we use the same notation for the zeta function and its meromorphic extension.

Thus, the number D is a real pole of $\zeta_{\mathcal{L}}$ and it is the Minkowski dimension of $\mathcal{L}$. This suggests to extend the notion of dimension to complex numbers by considering the poles of the meromorphic extension of $\zeta_{\mathcal{L}}$:

Definition 3 *The set of complex dimensions of a fractal string $\mathcal{L}$ contained in some region $R \subseteq \mathbb{C}$ is*

$$\mathcal{D}(R) = \{\omega \in R | \ \ \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}.$$

(See Definition 1.12 in [18] for a more general definition.) The relevance of considering complex dimensions is illustrated in particular by the following theorem :

Theorem 2 *[18] If a fractal string satisfies certain mild conditions, the following are equivalent:*

1. *$\mathcal{L}$ has only one complex dimension with real part equal to $D_{\mathcal{L}}$ and this pole is a simple pole.*
2. *$\partial\mathcal{L}$ is Minkowski measurable.*

See [18] for more details, in particular Theorem 8.15 and the discussion leading up to this theorem. More refined results exist, that allow for instance to characterize the asymptotic behavior of $V(\varepsilon)$ (as ε tends to 0) in terms of the residues of $\zeta_{\mathcal{L}}$.

Example: A simple example will best illuminate and motivate these definitions. The example we choose is the standard middle-1/3 Cantor subset of $[0, 1]$. See reference [18] pages 13-16 for a more complete discussion of this example. The lengths for this fractal string are

$$1/3, 1/9, 1/9, 1/27, 1/27, 1/27, 1/27, 1/81, \ldots, 1/81, 1/243, \ldots$$

with 2^{n-1} copies of 3^{-n} for $n \geq 1$. It is easy to see that

$$\zeta_{\mathcal{L}}(s) = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}},$$

so that $\zeta_{\mathcal{L}}$ has a meromorphic extension to all of $\mathbb{C}$. Thus, the poles of $\zeta_{\mathcal{L}}(s)$ are the points

$$\frac{\ln(2)}{\ln(3)} + \frac{2\pi i k}{\ln(3)}, \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots. \tag{1}$$

In this particular case, we have (see [18])

$$V(\epsilon) = 2\epsilon \cdot (2^n - 1) + (2/3)^n = (2\epsilon)^{1-D} \left((1/2)^{\{-\log_3(2\epsilon)\}} + (3/2)^{\{-\log_3(2\epsilon)\}} \right) - 2\epsilon,$$

where $n = [-\log_3(2\epsilon)]$ and $x = [x] + \{x\}$ is the decomposition of a positive real number into its integer and fractional parts and $D = \ln(2)/\ln(3)$ is the dimension of the Cantor set. This means that

$$V(\epsilon) = \epsilon^{1-D} G(\epsilon) - 2\epsilon \tag{2}$$

where G is a non-constant multiplicatively periodic function of multiplicative period $2\pi/\ln(3)$ (see [18], page 16). Notice the connection between the inter-pole spacing of $\zeta_{\mathcal{L}}(s)$ and the oscillations of $V(\epsilon)$ (that is, there is the quantity $2\pi/\ln(3)$ in equations (1) and (2)).

The primary purpose of this paper is to find the same connections between the poles of a zeta function of a fractal string and the behaviour of some relevant dimensional quantity (such as $V(\epsilon)$ in the example above) in other situations. In that view, in Section 2 we first propose a slight modification of the definition of a zeta function for a fractal string that allows us to extend the idea of using complex dimensions in more general settings, which are not easily accessible with the classical definition. In Section 3 we then use this modified zeta function to define multifractal zeta functions in both a discrete setting and a continuous setting. We prove (Propositions 3 and 6) that, in the case of a self-similar fractal measure, the critical exponents of convergence of these two multifractal zeta functions are equal to the negative of the Legendre transform of the multifractal spectrum of the measure. Moreover, we show (Proposition 5 and Corollary 2) that their poles allow to detect and measure the possible oscillatory behaviour of the continuous partition function, a quantity of interest in the Legendre approach to multifractal analysis. Our results thus parallel to a certain extent the ones relating the asymptotic behavior of $V(\varepsilon)$ to the poles of $\zeta_{\mathcal{L}}$ in the usual fractal case (as in the example above). Next in Section 4 we apply this modified zeta function to fractals in higher dimensions. We indicate (Propositions 7, 8, and 8) that for self-similar fractal in $\mathbb{R}^n$ the same connection between the poles of the (modified) zeta function and the oscillations of $V(\epsilon)$ holds as in the usual one-dimensional case.

Related recent work

A preliminary work on the extension of fractal strings to a multifractal setting has been considered in [32, 21, 27], using a somewhat different approach. More recently, independent work by Lapidus and Rock has been developed in [27, 32] (with [32] being a PhD thesis), which is more similar in spirit to our Section 3. In [27], the authors define a *partition zeta function* and use it to produce a multifractal zeta function. Their approach also starts with a sequence of partitions of a set, but then collects all the intervals on which the measure μ has a given "regularity" α and uses these intervals as the basis for a zeta function. Thus, a given family of partitions will determine at most countably many zeta functions (since it determines at most countably many regularity values). Looking at the abscissa of convergence σ as a function of the regularity one can then recover the usual multifractal spectrum, at least in the example of self-similar measures.

2 A modified zeta function for fractal strings

In this section we define a modification of the usual zeta function for fractal strings. This modification will allow us to define in a simple way a zeta function for fractals in $\mathbb{R}^n$ (and thus the beginnings of a formalism for higher-dimensional fractal strings) and a multifractal zeta function.

The usual definition of strings is based on adding the lengths of the *complement* of the set one wishes to study. Usually, this is done in a sequential way: One first removes the interval with largest length, and then goes on by decreasing order of lengths. This raises problems in the multifractal frame: Assume one wishes to define a zeta function that would characterize the subset E_α of points which have a given regularity. Contrarily to the geometric situations considered usually, the approximations E_α^n at each scale of E_α do not form a decreasing sequence, *i.e.* E_α^n is generally not in E_α^{n-1}. As a consequence, it is not clear how to measure the length that is removed when one passes from scale n to scale $n+1$. In contrast, the lengths of the approximations at each scale are perfectly well defined and easy to evaluate. We see that "adding the lengths of what remains" is easier than "adding the lengths of what is removed" in a multifractal setting. The same situation occurs when attempting to define a zeta function for a higher dimensional self-similar set. The problem is that the complement of such a set is rarely a disjoint union of "holes" as in the case of subsets of $\mathbb{R}$, thus summing the "lengths" of the components of the complement is not easily defined. However the construction of such a set gives a natural sequence of covers of the set (like the closed intervals in the construction of the standard 1/3 Cantor set) and these covers can easily be used to define a zeta function.

With this in mind, we propose an alternative definition of zeta functions for fractal strings that uses the lengths of the approximations at each scale, rather than the lengths that are removed at each scale. This might seem at first sight to lead to a very different object. However, we show that, in situations of interest (most notably in self-similar case, but also in more general cases as is seen in Proposition 1, below), the usual and alternative definitions will in fact yield "almost" the same zeta functions. In particular, they will share the same poles. The intuitive reason for this is that only the asymptotic behaviour of the lengths matters. This new definition is put to use in the present work to study both the multifractal behaviour of measures and higher dimensional geometric fractal objects.

Definition 4 *Let C be a bounded subset of $\mathbb{R}^d$, and let $\{\mathcal{C}_n\}$ be a sequence of covers of C with diameters tending to 0, i.e. each $\mathcal{C}_n$ is an at most countable collection of measurable subsets I_n^i of $\mathbb{R}^d$ such that $\lim_{n\to\infty}\sup_i |I_n^i| = 0$, where $|I|$ denotes the diameter of I. Then the* modified zeta function corresponding to $\mathcal{C}_n$ *is the function*

$$\zeta_c(s) = \sum_{n=1}^{\infty} \sum_{I\in\mathcal{C}_n} |I|^s. \tag{3}$$

The modified zeta function depends on the sequence of covers and not only on C. This is indeed a drawback but, in many cases of interest, there is a "natural" sequence associated to C. This is in particular the case for self-similar or self-affine sets.

The easy theorem below shows that, in the self-similar case, a simple relation holds between the usual and modified zeta functions:

Theorem 3 *(Self-similar strings in $\mathbb{R}$) Consider the self-similar fractal string generated by the IFS $(w_i)_{i=1,\dots,N}$, with $w_i(x) = r_i x + a_i$ on $[0,1]$ (see [7] for the basic theory of IFS) and with gaps of length $g_j, j = 1,2,\dots,K$ (that is, $[0,1] \setminus \cup_i w_i([0,1])$ is a union of open intervals with lengths g_j), and where we also assume that the open set condition is satisfied.*

Let ζ_c be the modified zeta function based on the natural sequence of covers defined by

$$\mathcal{C}_n = \{w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}([0,1]) : \sigma := (\sigma_1, \sigma_2, \cdots, \sigma_n) \in \{1,2,\dots,N\}^n\},$$

for $n = 0,1,\dots$. Then:

$$\zeta_{\mathcal{L}}(s) = g(s)(\zeta_c(s) + 1)$$

where $g(s)$ is analytic on $\mathbb{C}$. In particular, the two zeta functions have exactly the same poles.

Proof

The usual geometric zeta function of the corresponding fractal string satisfies the functional equation (obtained from the self-similarity of the fractal string)

$$\zeta_{\mathcal{L}}(s) = g_1^s + g_2^s + \cdots + g_K^s + r_1^s \zeta_{\mathcal{L}}(s) + r_2^s \zeta_{\mathcal{L}}(s) + \cdots + r_N^s \zeta_{\mathcal{L}}(s)$$

and thus

$$\zeta_{\mathcal{L}}(s) = \frac{\sum_{j=1}^{K} g_j^s}{1 - \sum_{i=1}^{N} r_i^s}.$$

For the natural sequence of covers, the modified zeta function satisfies

$$\zeta_c(s) = 1^s + r_1^s \zeta_c(s) + r_2^s \zeta_c(s) + \cdots + r_N^s \zeta_c(s)$$

and thus

$$\zeta_c(s) = \frac{1}{1 - \sum_{i=1}^{N} r_i^s}.$$

∎

In [18], generalized fractal strings are defined as measures on $(0,\infty)$: For a measure η such that $\eta(0,x_0) = 0$ for some $x_0 > 0$, the corresponding zeta function is given by:

$$\zeta_\eta(s) = \int_0^\infty x^{-s}\eta(dx).$$

For a usual string, setting $\eta = \sum_0^\infty \delta_{l_j^{-1}}$, where δ_u is the Dirac distribution centered at u, allows us to recover the usual $\zeta_{\mathcal{L}}$ function.

In our case, the measure η_c reads:

$$\eta_c = \sum_{n=1}^{\infty}\sum_{I\in\mathcal{C}_n} \delta_{|I|^{-1}}.$$

An easy computation shows that, in the self-similar case:

$$\eta_c = \sum_{(\sigma_1,\sigma_2,\cdots,\sigma_N)\in\mathbb{N}^N} \binom{\sigma_1+\sigma_2+\cdots+\sigma_N}{\sigma_1\sigma_2\cdots\sigma_N}\delta_{r_1^{-\sigma_1}r_2^{-\sigma_2}\cdots r_N^{-\sigma_N}}.$$

Note that this is exactly the same formula as formula 4.38 in [18] for the measure η associated to the classical $\zeta_{\mathcal{L}}$ function for a self-similar string where there is only one "gap" at the first stage.

There is another more general situation in which the two zeta functions also have the same poles. For this, we will need the following lemma:

Lemma 1 *Let $(l_n)_{n\geq1}$, $(\tilde{l}_n)_{n\geq1}$, $(\alpha_n)_{n\geq1}$ and $(\tilde{\alpha}_n)_{n\geq1}$ be four sequences of positive numbers such that*

$$\tilde{l}_n \leq C\ l_n, \quad \tilde{\alpha}_n \leq C\ \alpha_n,$$

where C is a constant. Denote the exponent of convergence of $\zeta(s) := \sum_{n=1}^\infty \alpha_n l_n^s$ by D and of $\tilde{\zeta}(s) := \sum_{n=1}^\infty \tilde{\alpha}_n \tilde{l}_n^s$ by $\tilde{D}$. Assume ζ and $\tilde{\zeta}$ both have a meromorphic extension in a neighborhood of the set $\{Re(z) \geq D\}$.
Then, $\tilde{D} \leq D$, and the set of poles of the meromorphic extension of $\tilde{\zeta}$ on the line $Re(z) = D$ is a subset of the set of poles of the meromorphic extension of $\zeta(s)$ on this line.

Proof: The assumptions give us that

$$\left|\sum_n \tilde{\alpha}_n \tilde{l}_n^s\right| \leq \sum_n \tilde{\alpha}_n |\tilde{l}_n^s| \leq C^{s+1}\sum_n \alpha_n |l_n^s|$$

which easily shows that $\tilde{D} \leq D$.

For the moment, we will use $\phi(s)$ for the meromorphic extension of $\zeta(s)$ and $\tilde{\phi}(s)$ for the meromorphic extension of $\tilde{\zeta}(s)$. By definition $\zeta(s) = \phi(s)$ for $Re(s) > D$ and $\tilde{\zeta}(s) = \tilde{\phi}(s)$ for $Re(s) > \tilde{D}$. Suppose that $D + iy$ is not a pole of ϕ. Then for any sequence $x_n + iy_n$ with $x_n > D$ and $x_n \to D$ and $y_n \to y$ we have that $\zeta(x_n + iy_n) = \phi(x_n + iy_n)$ converges to $\phi(D + iy)$, so in particular is a bounded sequence. However, this implies that $\tilde{\zeta}(x_n + iy_n)$ is also a bounded sequence and thus $D + iy$ is not a pole of $\tilde{\phi}(s)$ either.

Thus, at least on the line $Re(z) = D$, the poles of $\tilde{\zeta}$ are a subset of the poles of ζ. ∎

Corollary 1 *If both sequences $\tilde{\alpha}_n/\alpha_n$ and $\tilde{l}_n/l_n$ are bounded away from 0 and infinity, then $\tilde{D} = D$ and the set of poles of ζ and $\tilde{\zeta}$ on the line $Re(z) = D$ coincide.*

The following proposition is a simple consequence of the corollary.

Proposition 1 *Suppose that (l_n) is a sequence of lengths for a fractal string with $\limsup l_{2^n}/l_{2^{n+1}} < \infty$. Let C be the corresponding Cantor set. Suppose further that these lengths are arranged in such a way that for each n, there is a covering $\mathcal{P}_n$ of C consisting of 2^n closed intervals all of asymptotic size*

$$\frac{1}{2^n} \sum_{i \geq 2^n} l_i$$

and that

$$l_{2^n} \sim \frac{1}{2^n} \sum_{i \geq 2^n} l_i.$$

Then the poles of $\zeta_{\mathcal{L}}$ and ζ_c are the same, at least on the critical line with real part equal to D.

Proof: By assumption for each $I \in \mathcal{P}_n$ we have $|I| \sim l_{2^n}$ (as $n \to \infty$) and for each i with $2^n \leq i < 2^{n+1}$ we have $1 \leq l_n/l_i \leq M := \limsup l_{2^n}/l_{2^{n+1}}$. Thus there are $0 < a, b$ so that for $2^n \leq i < 2^{n+1}$ we have $a < l_i/|I| < b$ where $I \in \mathcal{P}_n$. Thus the two zeta functions

$$\sum_{n=1}^{\infty} \sum_{I \in \mathcal{P}_n} |I|^s \quad \text{and} \quad \sum_{n=1}^{\infty} \sum_{i=0}^{2^n} l_{2^n+i}^s$$

satisfy the conditions of the corollary. ∎

An example of a sequence of lengths which satisfies this is $l_n = 1/n^p, p > 1$. The condition that the lengths are arranged in a particular way will be generically satisfied (that is, almost every random arrangement would satisfy this condition - for a related question, see [11]).

As said above, the interest of ζ_c is that it is well defined as soon as a sequence of covers is given. Moreover, we see now that, if we are only interested in the poles and we are prepared to ignore multiplicative analytic perturbations, ζ_c and η_c give the same information as the usual fractal string and measure, at least in the self-similar case. In particular, since the formulas derived for the counting function and $\varepsilon-$neighborhoods of strings in chapters 5 and 8 of [18] rely on the expression of the measure η, we get that these formula are also valid if we use ζ_c and η_c.

However, more is true of this modified zeta function. By adapting the sequence of covers, the poles of the zeta function may be tailored to measure various dimensional concepts. For example, by using a "nearly optimal" sequence of $1/2^n$-covers, the modified zeta function ζ_c is sensitive to the Hausdorff dimension and Hausdorff measure of the boundary of the string.

In the following section, we thus adopt ζ_c as a relevant alternative definition of a zeta function. This point of view allows us to define and study in a simple way higher dimensional and multifractal zeta functions.

3 Multifractal zeta functions

3.1 Discrete setting

The so-called "Legendre approach" to multifractal analysis looks at the growth rate when δ tends to 0 of the quantity

$$S_\delta(q) = \sum_{I \in \mathcal{P}_\delta} \mu(I)^q$$

as a function of $q \in \mathbb{R}$, where μ is a measure on $[0,1]^d$ and $\mathcal{P}_\delta$ is a partition of $[0,1]^d$ into intervals of size δ. Roughly speaking, one defines a function $\tau(q)$ by the condition that $S_\delta(q) \simeq \delta^{\tau(q)}$, as $\delta \to 0$. More precisely:

$$\tau(q) = \liminf_{\delta \to 0} \frac{\log(S_\delta(q))}{\log(\delta)}. \tag{4}$$

In the general case of a non-uniform partition, the definition of τ is a little bit more involved, and goes as follow:

Let μ be a measure on $[0,1]^d$ and $(\mathcal{P}_n)$ be a sequence of partitions of $[0,1]^d$ into measurable sets with diameters tending to 0 when n tends to infinity. Fix a sequence (λ_n) of positive integers such that:

$$\forall \eta > 0, \ \sum_{n=1}^{\infty} \exp(-\eta \lambda_n) < \infty. \tag{5}$$

Typically, one may take $\lambda_n = \log(\mathrm{card}(\mathcal{P}_n))$, or $\lambda_n = \log(\inf_{I \in \mathcal{P}_n}(|I|))$, provided these quantities verify assumption (5). Define next the (discrete) partition function:

$$S_n(q, \beta) = \sum_{I \in \mathcal{P}_n^*} \mu(I)^q |I|^{-\beta}$$

where $\mathcal{P}_n^*$ consists of those intervals I in $\mathcal{P}_n$ such that $\mu(I) > 0$, and

$$S(q, \beta) = \limsup_{n \to \infty} \lambda_n^{-1} \log(S_n(q, \beta)).$$

Following [3], one shows that $S(q, \beta)$ is convex, non-increasing in q and non-decreasing in β. Further, defining:

$$\Omega = \{(q, \beta) : S(q, \beta) < 0\}$$

one shows that there exists a concave function τ such that:

$$int(\Omega) = \{(q, \beta) : \beta < \tau(q)\}$$

(τ does not have to be finite in general, but we assume here that this is the case, at least on an interval.)

One verifies that this definition coincides with the one in (4) when all the intervals in each partition have same length $\exp(-\lambda_n)$.

The interest of τ is that, under certain conditions (essentially the ones allowing us to apply Gartner-Ellis theorem [5]), the Legendre transform of τ is the large deviation multifractal spectrum of μ ([29]). Also, when the so-called strong multifractal formalism holds, this Legendre transform is also the Hausdorff multifractal spectrum of μ (see [1, 3, 7]). We define in this section a multifractal zeta function $\zeta(q, s)$ that generalizes this by allowing one to get more information than just the critical exponent τ.

A natural way to adapt the definition of the modified zeta function (3) so as to incorporate some information about μ is to weight each diameter $|I|$ in Definition 4 by the measure of I. This also amounts to summing the $S_n(q, \beta)$ over all n.

Definition 5 *Let μ be a measure on $[0, 1]^d$ and $(\mathcal{P}_n)$ be a sequence of partitions of $[0, 1]^d$ into measurable sets with diameters tending to 0 when n tends to infinity.*

The (discrete) multifractal zeta function for μ based on the sequence of partitions $\mathcal{P}_n$ *is the function*

$$\zeta(q, s) = \sum_n \sum_{I \in \mathcal{P}_n^*} \mu(I)^q |I|^s$$

for all $(q, s) \in \mathbb{R}^2$ such that the sum converges.

Since $\zeta(q, s)$ is a non increasing function of s, define:

$$\sigma(q) = \inf\{s : \zeta(q, s) < \infty\}.$$

Proposition 2 *For all q, $\sigma(q) = -\tau(q)$.*

Proof:

Let $s > \sigma(q)$. Then the sum defining $\zeta(q, s)$ converges. This implies that

$$\sum_{I \in \mathcal{P}_n^*} \mu(I)^q |I|^s \to 0$$

as $n \to \infty$, or: $\lim_{n\to\infty} \log(S_n(q, -s)) = -\infty$, and thus $S(q, -s) \leq 0$. By definition of τ, this means that $-s \leq \tau(q)$.

Conversely, let $-s < \tau(q)$. Then there exists $\eta > 0$ such that, for all sufficiently large n, $\frac{\log(S_n(q,-s))}{\lambda_n} < -\eta$. As a consequence, $S_n(q, -s)) < \exp(-\eta\lambda_n)$ and, by assumption (5), the sum defining $\zeta(q, s)$ converges. Thus $s \geq \sigma(q)$. ▮

Thus the exponent of convergence of $\zeta(q,s)$ for any fixed q is given by $-\tau(s)$. This parallels the fact that the Minkowski dimension is the exponent of convergence for usual fractal strings. To get more precise information, we proceed as in the case of fractal strings: We now take s to be a complex number, and note that $\zeta(q,s)$ is analytic for $Re(s) > \sigma(q)$. We assume that $\zeta(q,s)$ has a meromorphic extension to a neighborhood of the set $\{z : Re(z) \geq \sigma(q)\}$ and we abuse notation by denoting this meromorphic extension by $\zeta(q,s)$.

Clearly $\sigma(q)$ is a pole of $\zeta(q,s)$. The set of all poles is called the set of *complex dimensions.*

Definition 6 *Fix $q \in \mathbb{R}$. The set of* complex dimensions *of the multifractal zeta function $\zeta(q,s)$ is the set of poles of the meromorphic extension.*

Example 1 *(Self-similar probability measure)*

Choose N real numbers r_i with $\sum_i |r_i| \leq 1$ and $0 < p_i$ with $\sum_i p_i = 1$. Let the N contractions $w_i(x) = r_i x + a_i$ satisfy the OSC (open set condition) and $w_i([0,1]) \subset [0,1]$. Then this IFS uniquely defines a probability measure μ on $[0,1]$ which satisfies the self-similarity equation (see, *e.g.* [7]):

$$\mu(B) = \sum_i p_i \mu(w_i^{-1}(B)), \qquad \text{for all Borel } B \subset [0,1]. \tag{6}$$

We use the IFS maps to define our sequence of partitions. That is, we let $\mathcal{P}_n$ be the partition consisting of the sets

$$\{w_{i_1} \circ w_{i_2} \circ \cdots \circ w_{i_n}([0,1]) : i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, N\}\}$$

for each n. If the sets $w_i([0,1])$ do not form a partition of $[0,1]$, then we will need to include some more intervals in each $\mathcal{P}_n$. However, these "extra" intervals will not influence the values of $\zeta(q,s)$ since the μ measure of any of these intervals is always zero.

Clearly by construction $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$. It is straightforward that

$$\mu(w_{i_1} \circ w_{i_2} \circ \cdots \circ w_{i_n}([0,1])) = p_{i_1} p_{i_2} \cdots p_{i_n}$$

by the self-similarity equation (6).

For this self-similar fractal measure, we have the multifractal zeta function

$$\begin{aligned}
\zeta(q,s) &= \sum_n \sum_{I \in \mathcal{P}_n} \mu(I)^q |I|^s \\
&= \sum_n \sum_{k_1+k_2+\cdots+k_N=n} \left(p_1^{k_1} p_2^{k_2} \cdots p_N^{k_N}\right)^q \left(r_1^{k_1} r_2^{k_2} \cdots r_N^{k_N}\right)^s \\
&= \sum_n (p_1^q r_1^s + p_2^q r_2^s + \cdots + p_N^q r_N^s)^n \\
&= \frac{1}{1 - \sum_i p_i^q r_i^s}.
\end{aligned}$$

There is another way to see this formula. The self-similarity equation (6) translates into a functional equation for $\zeta(s,\beta)$:

$$\zeta(q,s) = 1 + \sum_i p_i^q r_i^s \zeta(q,s)$$

from which we easily get the same result.

By definition of $\sigma(q)$ being the abscissa of convergence of $\zeta(q,s)$, we see that, for such a self-similar measure, $\tau(q) = -\sigma(q)$ is defined by the implicit equation:

$$\sum_i p_i^q r_i^{-\tau(q)} = 1. \tag{7}$$

This expression for τ is well known. Its Legendre transform is the Hausdorff and Large Deviation multifractal spectrum of μ (see chapter 11 of [8]).

From a simple modification of the analysis in Section 2.5 of [18], we have:

Proposition 3 *For the self-similar measure of example 1, one has:*

$$\zeta(q,s) = \frac{1}{1 - \sum_i p_i^q r_i^s}.$$

As a consequence, the abscissa of convergence of $\zeta(q,s)$ is $-\tau(q)$, and:

- *If $(-\log(r_1), \cdots, -\log(r_N))$ is non-arithmetic, $\zeta(q,s)$ has only one pole with real part $-\tau(q)$.*
- *If $(-\log(r_1), \cdots, -\log(r_N))$ is υ-arithmetic, $\zeta(q,s)$ has an infinite number of poles with real part $-\tau(q)$, whose imaginary parts are of the form $\frac{2k\pi}{-\log(\upsilon)}$, $k \in \mathbb{Z}$.*

At this point, it is not obvious how the dichotomy arithmetic/non-arithmetic of $\zeta(q,s)$ translates into properties of μ. We shall see this more clearly in the continuous setting, developed in the next paragraphs.

3.2 Continuous setting

A continuous analogue of τ may be defined as follows. For simplicity, we restrict here to the case of subsets of $\mathbb{R}$, *i.e.* μ is a measure on $[0,1]$.

Definition 7 *For $q, \beta \in \mathbb{R}$ and $r > 0$, define the continuous partition function:*

$$Z_r(q,\beta) = r^{-1-\beta} \int_0^{1-r} \mu([x, x+r])^q dx,$$

with the convention that $0^q = 0$ for all $q \in \mathbb{R}$. For any $q \in \mathbb{R}$, define $\tau^{ct}(q) = \sup\{\beta : \lim_{r\to 0} Z_r(q,\beta) = 0\}$.

Note: Some texts (*e.g.* [8]) define $Z_r(q,\beta)$ as follows:

$$Z_r(q,\beta) = r^{-\beta} \int_0^{1-r} \mu([x, x+r])^{q-1} d\mu(x).$$

Definition 7 yields a τ function more adapted to our needs.

Definition 8 *Let (ρ_n) be a sequence of real numbers decreasing to 0. The continuous multifractal zeta function is defined as:*

$$\zeta^{ct}(q,s) = \sum_{n=1}^{\infty} \rho_n^{s-1} \int_0^{1-\rho_n} \mu([x, x+\rho_n])^q dx,$$

for all $(q,s) \in \mathbb{R}^2$ such that the series converges. Define also:

$$\sigma^{ct}(q) = \inf\{s : \zeta^{ct}(q,s) < \infty\}.$$

Essentially the same proof as in the discrete setting (Proposition 2) allows us to show the following:

Proposition 4 *For any sequence (ρ_n) of real numbers decreasing to 0 and any q: $\sigma^{ct}(q) \geq -\tau^{ct}(q)$. If the sequence (ρ_n) is such that $\sum \rho_n^\varepsilon < \infty$ for all $\varepsilon > 0$, then $\sigma^{ct}(q) = -\tau^{ct}(q)$.*

As usual, we now let s be a complex number, and assume that $\zeta^{ct}(q,s)$ has a meromorphic extension to a neighborhood of the set $\{z : Re(z) \geq \sigma^{ct}(q)\}$, still denoted $\zeta^{ct}(q,s)$. The set of all of its poles is called the set of *complex dimensions*.

In the next paragraph, we perform an analysis of the complex dimensions of a self-similar measure, and show that they are related to the limiting behaviour of $Z_r(q,\beta)$.

3.3 Continuous complex dimensions of a self-similar measure

We restrict for simplicity to the case of measures on $\mathbb{R}$ and consider a self-similar measure as in example 1.

As a warm-up, we consider a simple situation where we have only two maps $w_1(x) = \rho x$ and $w_2(x) = (1-\rho)x + \rho$, $\rho \in (0,1)$, associated to the probabilities (p_1, p_2). The measure μ verifies the self-similarity identity valid for all $x \in [0,1]$ and $r > 0$:

$$\mu([x, x+r]) = p_1 \mu\left(\left[\frac{x}{\rho}, \frac{x+r}{\rho}\right]\right) + p_2 \mu\left(\left[\frac{x-\rho}{1-\rho}, \frac{x+r-\rho}{1-\rho}\right]\right).$$

- When $x \in [0, \rho - r]$, $\left[\frac{x}{\rho}, \frac{x+r}{\rho}\right] \subset [0,1]$, while $\left[\frac{x-\rho}{1-\rho}, \frac{x+r-\rho}{1-\rho}\right] \bigcap \ (0,1) = \emptyset$.
- When $x \in [\rho, 1-r]$, $\left[\frac{x}{\rho}, \frac{x+r}{\rho}\right] \bigcap \ (0,1) = \emptyset$, while $\left[\frac{x-\rho}{1-\rho}, \frac{x+r-\rho}{1-\rho}\right] \subset [0,1]$.

As a consequence, for all $r > 0$:

$$\begin{aligned}
r^{\beta+1} Z_r(q,\beta) &= \int_0^{1-r} \mu([x,x+r])^q dx \\
&= p_1^q \int_0^{\rho-r} \mu\left(\left[\frac{x}{\rho}, \frac{x+r}{\rho}\right]\right)^q dx + \int_{\rho-r}^{\rho} \mu([x,x+r])^q dx + \\
&\quad p_2^q \int_{\rho}^{1-r} \mu\left(\left[\frac{x-\rho}{1-\rho}, \frac{x+r-\rho}{1-\rho}\right]\right)^q dx \\
&= \rho p_1^q \int_0^{1-r/\rho} \mu\left(\left[x, x+\frac{r}{\rho}\right]\right)^q dx + \int_{\rho-r}^{\rho} \mu([x,x+r])^q dx + \\
&\quad (1-\rho) p_2^q \int_0^{1-r/(1-\rho)} \mu\left(\left[x, x+\frac{r}{1-\rho}\right]\right)^q dx \\
&= \rho p_1^q (\frac{r}{\rho})^{\beta+1} Z_{\frac{r}{\rho}}(q,\beta) + \int_{\rho-r}^{\rho} \mu([x,x+r])^q dx + \\
&\quad (1-\rho) p_2^q (\frac{r}{1-\rho})^{\beta+1} Z_{\frac{r}{1-\rho}}(q,\beta)
\end{aligned}$$

Note that $Z_r(q,\beta) = 0$ for $r > 1$. Writing $h(r) = r^{-\beta-1} \int_{\rho-r}^{\rho} \mu([x,x+r])^q dx$, we get:

$$Z_r(q,\beta) = p_1^q \rho^{-\beta} Z_{\frac{r}{\rho}}(q,\beta) + p_2^q (1-\rho)^{-\beta} Z_{\frac{r}{1-\rho}}(q,\beta) + h(r). \tag{8}$$

The identity (8) allows us to make use of the now classical application of renewal theory to the study of self-similar fractals, pioneered in [13] (see [8] for an excellent introduction to this topic). Following *e.g.* [8], chapter 7, set:

$$r = \exp(-t), \quad f(t) = Z_{\exp(-t)}(q,\beta), \quad g(t) = h(\exp(-t)).$$

Note that g is a continuous function. Using g, (8) now reads:

$$f(t) = p_1^q \rho^{-\beta} f(t + \log(\rho)) + p_2^q (1-\rho)^{-\beta} f(t + \log(1-\rho)) + g(t).$$

Fix now $\beta = \beta(q)$ to be the unique value verifying $p_1^q \rho^{-\beta} + p_2^q (1-\rho)^{-\beta} = 1$ (remark that this value is really $\tau(q)$ defined in the discrete setting of section 3.1 for a self-similar measure). Note that this implies that $p_1^q \rho^{-\beta} < 1$, i.e., $\beta(q) < q \frac{\log(p_1)}{\log(\rho)}$ for all q. Likewise, $\beta(q) < q \frac{\log(p_2)}{\log(1-\rho)}$.

Assume without loss of generality that $p_2^{\frac{\log(r)}{\log(1-\rho)}} \leq p_1^{\frac{\log(r)}{\log(\rho)}}$ (note that if this is true for some $r \in (0,1)$, it is true for all such r). The measure μ verifies, for all x, r:

$$p_2^{\frac{\log(r)}{\log(1-\rho)}} \leq \mu([x,x+r]) \leq p_1^{\frac{\log(r)}{\log(\rho)}}.$$

To see this, consider an interval of the form

$$I_\sigma = w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}([0,1]), \quad \sigma_i \in \{1,2\}.$$

Let $m = \#\{i : \sigma_i = 1\}$. Then we have

$$\mu(I_\sigma) = p_1^m \ p_2^{n-m} \quad \text{and} \quad |I_\sigma| = \rho^m \ (1-\rho)^{n-m}.$$

Thus

$$\frac{\log(p_2)}{\log(1-\rho)} \leq \frac{\log(\mu(I_\sigma))}{\log(|I_\sigma|)} = \frac{m\log(p_1) + (n-m)\log(p_2)}{m\log(\rho) + (n-m)\log(1-\rho)} \leq \frac{\log(p_1)}{\log(\rho)},$$

which is equivalent to the inequality we wished to show for intervals of the special form above. For a general interval, we use a standard approximation procedure.

As a consequence, for $q \geq 0$:

$$\begin{aligned} h(r) &\leq r^{-\beta(q)-1} \int_{\rho-r}^{\rho} p_1^{q\frac{\log(r)}{\log(\rho)}} \, dx \\ &\leq r^{-\beta(q)+q\frac{\log(p_1)}{\log(\rho)}}. \end{aligned}$$

This means that $g(t) \leq \exp(-\gamma t)$ with $\gamma = -\beta(q) + q\frac{\log(p_1)}{\log(\rho)} > 0$ for all fixed $q \geq 0$.

For $q < 0$, one has similarly:

$$\begin{aligned} h(r) &\leq r^{-\beta(q)-1} \int_{\rho-r}^{\rho} p_2^{q\frac{\log(r)}{\log(1-\rho)}} \, dx \\ &\leq r^{-\beta(q)+q\frac{\log(p_2)}{\log(1-\rho)}}. \end{aligned}$$

Thus $g(t) \leq \exp(-\gamma t)$ with $\gamma = -\beta(q) + q\frac{\log(p_2)}{\log(1-\rho)} > 0$ for all fixed $q \geq 0$.

This bound is sufficient to apply the renewal theorem for $t \geq 0$. However, it does not work for $t < 0$ as the example of Lebesgue measure shows. This corresponds to setting $\rho = p_1 = p_2 = 1/2$. In this case, $\beta(q) = q - 1$ so $h(r) = r^{-q} \int_{1/2-r}^{1/2} r^q \, dx = r$ and thus $g(t) = \exp(-t)$, which tends to infinity as $t \to -\infty$.

We thus apply the same trick as in [8], proposition 7.4. That is, we define

$$\tilde{f}(t) = 0 \text{ for } t < 0, \quad \tilde{f}(t) = f(t) \text{ for } t \geq 0,$$

and

$$\tilde{g}(t) = 0 \text{ for } t < 0,$$

and

$$\tilde{g}(t) = g(t) - p_1^q \rho^{-\beta} f(t+\log(\rho))\mathbb{1}(\rho < \exp(-t)) - p_2^q (1-\rho)^{-\beta} f(t+\log(1-\rho))\mathbb{1}(1-\rho < \exp(-t)) \text{ for } t \geq 0.$$

The renewal theorem may now be applied to yield that $\tilde{f}(t)$ tends to a constant when t tends to infinity if $\frac{\log(\rho)}{\log(1-\rho)}$ is not a rational number, and is equivalent to a periodic function otherwise. In terms of $Z_r(q,\beta)$, this reads:

$$\lim_{r\to 0} Z_r(q,\beta) = C \text{ if } (-\log(\rho), -\log(1-\rho)) \text{ is non-arithmetic,}$$

and

$$Z_r(q,\beta) \sim_{r\to 0} m(-\log(r)) \text{ if } (-\log(\rho), -\log(1-\rho)) \text{ is } \upsilon-\text{arithmetic},$$

where m is a positive periodic function of period υ.

In general, one has the following:

Proposition 5 *Let μ be a self-similar measure on $[0,1]$ satisfying the open set condition as in example 1. Define $\tau(q)$ as in equation (7). Then:*

$$\lim_{r\to 0} Z_r(q,\tau(q)) = C \text{ if } (-\log(r_1),\cdots,-\log(r_N)) \text{ is non-arithmetic},$$

and

$$Z_r(q,\tau(q)) \sim_{r\to 0} m(-\log(r)) \text{ if } (-\log(r_1),\cdots,-\log(r_N)) \text{ is } \upsilon-\text{arithmetic},$$

where m is a positive periodic function of period υ.

Proof: The proof is merely a rewriting of what we did in the simple case above. Starting from the self-similarity equation, one gets:

$$r^{\beta+1} Z_r(q,\beta) = \sum_{i=1}^{N} p_i^q \int_{a_i}^{a_i+r_i-r} \mu\left(\left[\frac{x-a_i}{r_i}, \frac{x+r-a_i}{r_i}\right]\right)^q dx + I(r),$$

where $I(r) = \int_{A(r)} \mu([x,x+r])^q dx$ and:

$$A(r) = \{x, \exists i : [x,x+r] \cap [a_i, a_i+r_i-r] \cap [a_{i+1}, a_{i+1}+r_{i+1}-r] \neq \emptyset\}.$$

Note that $A(r)$ is made of at most $N-1$ intervals each of which is of length at most r. By change of variables, we get

$$Z_r(q,\beta) = \sum_{i=1}^{N} p_i^q r_i^{-\beta} Z_{\frac{r}{r_i}}(q,\beta) + h(r)$$

with $h(r) = r^{-1-\beta} I(r)$.

Take now $\beta = \tau(q)$, so that $\sum_{i=1}^{N} p_i^q r_i^{-\tau(q)} = 1$. As above, one shows that $g(t) := h(\exp(-t)) \leq (N-1)\exp(-\gamma t)$ for some $\gamma > 0$, a bound valid for any $t \geq 0$, and any $t < 0$ in the case where $q > 0$. To apply the renewal theorem, one then defines functions $\tilde{f}$ and $\tilde{g}$ as above. ∎

We would like to link this result to the behaviour of the continuous multifractal zeta function $\zeta^{ct}(q,s) = \sum_{n=1}^{\infty} Z_{\rho_n}(q,-s)$. This can be done under the assumption that, for all $\varepsilon > 0, \sum \rho_n^{\varepsilon} < \infty$. For this, we will need the result from Lemma 1.

Proposition 6 *Let μ be a self-similar measure on $[0,1]$ satisfying the open set condition as in example 1. Define $\tau(q)$ as in equation (7). Let $(\rho_n)_{n\geq 1}$ be a decreasing sequence of real numbers tending to 0 such that $\sum \rho_n^\varepsilon < \infty$ for all $\varepsilon > 0$. Then, for each q, the abscissa of convergence of $\zeta^{ct}(q,s)$ is equal to $-\tau(q)$. In addition:*

- *If $(-\log(r_1), \cdots, -\log(r_N))$ is non-arithmetic, the poles of $\zeta^{ct}(q,s)$ and of (the meromorphic extension of) $\sum \rho_n^s$ coincide on the line $Re(s) = -\tau(q)$.*
- *If $(-\log(r_1), \cdots, -\log(r_N))$ is υ-arithmetic, the poles of $\zeta^{ct}(q,s)$ on the line $Re(s) = -\tau(q)$ are a subset of the ones of (the meromorphic extension of) $\sum \rho_n^s$.*

Proof: Let $\varepsilon = s + \tau(q)$. One has:

$$\zeta^{ct}(q,s) = \sum_{n=1}^{\infty} \rho_n^\varepsilon Z_{\rho_n}(q, \tau(q)).$$

In the non-arithmetic case, applying the remark after lemma 1 with $l_n = \tilde{l}_n = \rho_n$, $\alpha_n = 1$, $\tilde{\alpha}_n = Z_{\rho_n}(q, \tau(q))$ (which tends to a constant when n tends to infinity by proposition 5), we get the result.

In the arithmetic case, take instead $\tilde{\alpha}_n = m(-\log(\rho_n))$ (with the notations of proposition 5) and the others the same. Since m is a continuous function, m is bounded. Lemma 1 and proposition 4 imply the result.

∎

In the case where one takes a sequence $(\rho_n)_{n\geq 1}$ adapted in a natural way to the self-similar measure, one can get a more precise result:

Corollary 2 *Let μ be a self-similar measure on $[0,1]$ satisfying the open set condition as in example 1. Define $(\rho_n)_{n\geq 1}$ as the sequence of all possible products $r_1^{k_1} r_2^{k_2} \ldots r_N^{k_N}$ where $k_i \in \{0, 1, \ldots\}$, arranged in non-increasing order. Then, for each q, the abscissa of convergence of $\zeta^{ct}(q,s)$ is equal to $-\tau(q)$ and the poles of $\zeta^{ct}(q,s)$ and of $\sum \rho_n^s$ coincide on the line $Re(s) = -\tau(q)$. In particular:*

- *If $(-\log(r_1), \cdots, -\log(r_N))$ is non-arithmetic, $\zeta(q,s)$ has only one pole with real part $-\tau(q)$.*
- *If $(-\log(r_1), \cdots, -\log(r_N))$ is υ-arithmetic, $\zeta(q,s)$ has an infinite number of poles with real part $-\tau(q)$, whose imaginary part are of the form $\frac{2k\pi}{-\log(\upsilon)}$.*

Proof: First note that the zeta function $\zeta(s) = \sum \rho_n^s$ corresponds to a self-similar fractal string so the poles of this zeta function satisfy the two conclusions of the corollary (see, *e.g.* Theorem 2.17 in [18]).

For the non-arithmetic case, proposition 6 gives the coincidence of the poles of ζ^{ct} and ζ. In the arithmetic case, we have by definition that for all i, $r_i = r^{k_i}$, $k_i \in \mathbb{N}$, where $r = \exp(-\upsilon)$. Therefore,

$$\log(\rho_n) = \log(\prod r_{i_j}) = (\sum k_{i_j}) \log(r). \tag{9}$$

Thus:

$$m(-\log(\rho_n)) = m\left((\sum k_{i_j})\upsilon\right) = m(\upsilon),$$

since m is υ-periodic. ∎

We see that the results in the continuous frame are exactly the same as the ones in the discrete case (proposition 3), at least to the right of and on the vertical line with abscissa $-\tau(q)$. However, there is a significant difference between the two settings: In the continuous frame, we can relate the fact that $\zeta^{ct}(q,s)$ has or does not have only one pole with real part $-\tau(q)$ to the "oscillatory" behaviour of some quantity, namely $Z_r(q,\tau(q))$. There does not seem to be a quantity corresponding to $Z_r(q,\tau(q))$ in the discrete frame. Indeed, $\sum_{I\in\mathcal{P}_n} \mu(I)^q |I|^{-\tau(q)} = \left(p_1^q r_1^{-\tau(q)} + p_2^q r_2^{-\tau(q)} + \cdots + p_N^q r_N^{-\tau(q)}\right)^n = 1$ for all n, and thus never oscillates. The poles of the continuous and of the discrete multifractal zeta functions both give information on $Z_r(q,\tau(q))$.

Corollary 2 gives a characterization of the oscillatory behaviour of the continuous partition function in terms of the poles of the multifractal zeta function in the case of self-similar measures. This is a natural generalization of the corresponding link between $V(\varepsilon)$ and the usual zeta function for fractal strings. Further work should focus on extending this result to a more general frame.

4 Zeta functions for fractals in $\mathbb{R}^n$

The modified zeta function is defined in any dimension. We study in this section the particular case of a self-similar set $E \subset \mathbb{R}^n$ generated by an IFS satisfying the open set condition. The strong connection between ζ_c and $\zeta_{\mathcal{L}}$ in one dimension and in the self-similar case makes it plausible for the poles of ζ_c to give relevant information, as will shown to be the case.

Our main aim in this section is to show how our modified zeta function (based on a natural sequence of covers) provides a language in which the standard results from the theory of fractal strings can (hopefully all) be transfered to the setting of subsets of $\mathbb{R}^n$.

Previous work has defined geometric zeta functions for some classes of higher-dimensional fractals. In particular, for *fractal sprays* in [18] and a *canonical tiling* associated with a self-similar set in [24]. The ideas in the paper [24] are very interesting and, for a wide class of situations, provide a natural generalization of fractal strings to higher dimensions. The zeta function constructed in [24] provides information about the volume of tubular neighborhoods of the fractal and thus leads to the Minkowski dimension. Our construction, by contrast, can lead to information about any dimension (or measure) defined by sequences of coverings, in particular the Hausdorff dimension and measure. The drawback is that one must explicitly construct the desired cover. However, in the case of a fractal defined by an IFS, the sequence of covers is canonically given.

Recall that, for a self-similar set E which satisfies the strong separation condition, the Minkowski dimension is the real number D satisfying $\sum_i r_i^D = 1$. The proof of the following is analogous to the situation in one dimension (see, for example, Theorem 2.17 in [18] for comparison) and thus we leave it out. The interest is that our zeta function provides the same results for a large class of self-similar subsets of $\mathbb{R}^n$. In fact, the three results in this section are all modifications of known results which are recast in our setting, the interest being to point towards a possible theory of geometric zeta functions for sets in $\mathbb{R}^n$.

Proposition 7 *(self-similar fractals in $\mathbb{R}^n$) Consider the self-similar fractal generated by the IFS $(w_i)_{i=1\ldots N}$, where the w_i are similarities on $\mathbb{R}^n$ with ratio $r_i \in (0,1)$ and satisfying the strong separation condition. Let ζ_c be the modified zeta function based on the natural sequence of covers defined by*

$$\mathcal{C}_n = \{w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}([0,1]^d) : \sigma \in \{1,2,\ldots,N\}^n\}, \quad n = 0,1,2,\ldots.$$

Then:

$$\zeta_c(s) = \frac{d^{s/2}}{1 - \sum_i r_i^s}.$$

In particular, the abscissa of convergence of $\zeta_c(s)$ is equal to D, the Minkowski dimension of E. In addition:

- *If $(-\log(r_1), \cdots, -\log(r_N))$ is non-arithmetic, $\zeta_c(s)$ has only one (simple) pole with real part D.*
- *If $(-\log(r_1), \cdots, -\log(r_N))$ is υ-arithmetic, $\zeta_c(s)$ has an infinite number of poles with real part D, whose imaginary part are of the form $\frac{2k\pi}{-\log(\upsilon)}$.*

A major interest of zeta functions associated to fractal strings is that they allow us to obtain, under certain assumptions, precise asymptotic formulas for various counting functions (see [18]), for instance the number of lengths of the string larger than ϵ as $\epsilon \to 0$. These formulas hold in one dimension. Our aim here is not to prove that these general formulas hold in higher dimensions, but only to check that, in the self-similar case, the complex dimensions of the d-dimensional $\zeta_c(s)$ indeed give the same type of information as in one dimension. We again use renewal theory. Recall the following result ([8], proposition 7.4):

Proposition 8 *Let E be the attractor of the IFS $(w_i)_{i=1\ldots N}$, where the w_i are similarities on $\mathbb{R}^n$ with ratio $r_i \in (0,1)$ and satisfying the strong separation condition. Let $N(r)$ denote the least number of sets of diameter r that can cover E. Then:*

- *If $\{-\log(r_1), \cdots, -\log(r_N)\}$ is non-arithmetic, there exists $c > 0$ such that $N(r) \sim cr^{-D}$ when r tends to 0.*

- *If $\{-\log(r_1), \cdots, -\log(r_N)\}$ is τ-arithmetic, there exists a positive periodic function p with period τ such that $N(r) \sim r^{-D} p(-\log(r))$ when r tends to 0.*

The next result makes explicit the relationship between the poles of $\zeta_c(s)$ on the critical line and the behaviour of $V(r)$ as $r \to 0$ (and, hence, gives information about the Minkowski measurability of the associated self-similar fractal set). The strong separation condtion implies that the fractal is homeomorphic to a Cantor set. Thus, instead of the one-sided tubular neighborhood (as before) we must consider the volume of the r-dilation of the set E ($r > 0$) given by

$$V(r) = \mathrm{vol}_n\{x : d(x, E) < r\}.$$

The following theorem is a simple modification of Prop 7.4 in [8]. It also (essentially) appeared as Theorem 2.3 in [9]. Note however that, thanks to our modified definition, the proof is here quite simple.

Proposition 9 *Let E be the attractor of the IFS $(w_i)_{i=1\dots N}$, where the w_i are similarities on $\mathbb{R}^n$ with ratio $r_i \in (0, 1)$ and satisfying the strong separation condition. Let $V(r)$ denote the n-dimensional volume of the r-dilation of E. Then, with D the Minkowski dimension of E,*

- *If $\zeta_c(s)$ has only one (real) pole with real part D, then there exists a $c > 0$ such that $V(r) \sim cr^{n-D}$ as $r \to 0$.*
- *If $\zeta_c(s)$ has an infinite number of poles with real part D of the form $\frac{2k\pi}{-\log(\nu)}$, then there is a positive periodic function p with period ν so that $V(r) \sim r^{n-D} p(-\log(r))$ as $r \to 0$.*

Proof: Because E is a self-similar set satisfying the strong separation condition, by Proposition 7 the two cases (conditions on the poles of ζ_c) are equivalent to $(-\log(r_1), \cdots, -\log(r_N))$ being either non-arithmetic or ν-arithmetic for some $\nu > 0$.

We can modify the proof of Prop 7.4 in [8] to show that either $V(r) \sim cr^{n-D}$ or $V(r) \sim r^{n-D} p(-\log(r))$ as $r \to 0$. The only real changes required are that we obtain for $0 < r < d := \min_{i \neq j} \mathrm{dist}(w_i(E), w_j(E))$ (notation as in pages 123-125 of [8])

$$V(r) = \sum_i r_i^n V(r/r_i) - h(r)$$

and thus we have to use the definitions $f(t) = e^{-(D-n)t}\, V(e^{-t})$ and $g(t) = e^{-(D-n)t}\, h(e^{-t})$. However, we then have the required link between the poles of ζ_c and the behaviour of $V(r)$ as $r \to 0$. ∎

Notice that proposition 9 implies that E is Minkowski measurable in the non-arithmetic case but makes no direct claims about Minkowski measurability in the ν-arithmetic case, which would require further information about the periodic function p (in particular, whether p is constant).

Propositions 7,8 and 9 show that we recover typical results of the theory of fractal strings about the dichotomy between the arithmetic/non-arithmetic cases, and, in the arithmetic case, the correct period of oscillation for both $N(r)$ and $V(r)$. Furthermore, for a self-similar fractal E satisfying the strong separation condition, we also get a relationship between poles on the critical line and the Minkowski measurability of E. This hints that ζ_c may be a meaningful path to a higher dimensional generalization of fractal strings. The general situation ought to be more complex, though, since, for instance, a direct transposition of the computations above does not lead to meaningful results in the case of a self-affine set.

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References

[1] M. Arbeiter and N. Patzschke. Random self-similar multifractals. *Math. Nachr.*, 181:5–42, 1996.

[2] A. Besicovitch and S. Taylor On the complementary intervals of a linear closed set of zero Lebesgue measure *J. London Math. Soc.* 29:449-459, 1954.

[3] G. Brown, G. Michon, and J. Peyrière. On the multifractal analysis of measures. *J. Statist. Phys.*, 66(3-4):775–790, 1992.

[4] C. Cabrelli, F. Mendivil, U. Molter, R. Shonkwiler On the Hausdorff h-Measure of Cantor Sets *Pacific Journal of Mathematics*, 217(1)45-59, 2004.

[5] R.S. Ellis, Large Deviations for a General Class of Random Vectors, *Ann. Prob.*, 12(1):1–12, 1984.

[6] K. E. Ellis, M. L. Lapidus, M. C. MacKenzie, J. A. Rock, Partition zeta functions, multifractal spectra, and tapestries of complex dimensions, eprint arXiv:1007.1467 [math.ph], 2010

[7] K. Falconer, *Fractal Geometry - Mathematical Foundations and Applications*, John Wiley, Second Edition, 2003,.

[8] K. Falconer, *Techniques in Fractal Geometry*, John Wiley, 1997.

[9] D. Gatzouras, Lacunarity of self-similar and stochastically self-similar sets. *Trans. Amer. Math. Soc.* **352**(5): 1953-1983, 2000.

[10] B.M. Hambly and M. L. Lapidus, Random Fractal Strings: Their Zeta Functions, Complex Dimensions and Spectral Asymptotics, *Trans Amer. Math. Soc.* No.1, 358 (2006), 285-314.

[11] J. Hawkes, Random re-orderings of intervals complementary to a linear set, *Q. J. Math*, 35, (1984), 1650172.

[12] C. Q. He and M. L. Lapidus, Generalized Minkowski content, spectrum of fractal drums, fractal strings and the Riemann zeta-function, *Memoirs Amer. Math. Soc.*, 608, 127 (1997), 197.

[13] S.P. Lalley, The packing and covering functions of some self-similar fractals, *Indiana Univ. Math. J.*, 37, (1988), 699709.

[14] M. L. Lapidus, Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture, *Trans. Amer. Math. Soc.* 325 (1991), 465529.

[15] M. L. Lapidus, Spectral and fractal geometry: From the Weyl-Berry conjecture for the vibrations of fractal drums to the Riemann zeta-function, in: *Differential Equations and Mathematical Physics* (C. Bennewitz, ed.), Proc. Fourth UAB Internat. Conf. (Birmingham, March 1990), Academic Press, New York, 1992, 151182.

[16] M. L. Lapidus, Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media, and the Weyl-Berry conjecture, in: *Ordinary and Partial Differential Equations* (B. D. Sleeman and R. J. Jarvis, eds.), vol. IV, Proc. Twelfth Internat. Conf. (Dundee, Scotland, UK, June 1992), Pitman Research Notes in Math. Series, vol. 289, Longman Scientific and Technical, London, 1993, 126209.

[17] M. L. Lapidus, Fractals and vibrations: Can you hear the shape of a fractal drum? *Fractals* 3, No. 4 (1995), 725736. (Special issue in honor of Benoit B. Mandelbrot's 70th birthday.)

[18] M. L. Lapidus and M. van Frankenhuijsen, *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings*, Springer, New York, 2006.

[19] M. L. Lapidus and M. van Frankenhuijsen, A prime orbit theorem for self-similar flows and Diophantine approximation, *Contemporary Mathematics* 290 (2001), 113138.

[20] M. L. Lapidus and M. van Frankenhuijsen, Complex dimensions of self-similar fractal strings and Diophantine approximation, *J. Experimental Mathematics*, No. 1, 42 (2003), 4369.

[21] M. L. Lapidus, J. Lévy Véhel and J. A. Rock, Fractal strings and multifractal zeta functions, *Letters in Mathematical Physics* (special issue dedicated to Moshe Flato), 2009, in press. (e-print: ArXiv:math-ph/0610015.)

[22] M. L. Lapidus and H. Maier, The Riemann hypothesis and inverse spectral problems for fractal strings, *J. London Math. Soc.* (2) 52 (1995), 1534.

[23] M. L. Lapidus and E.P.J Pearse, A Tube formula for the Koch snowflake curve with applications to complex dimensions, *J. London Math. Soc.* (2) 74 (2006), pp. 397-414.

[24] M. L. Lapidus and E.P.J. Pearse, Tube formulas and complex dimensions of self-similar tilings, to appear in *Acta Appl. Math.*, arXiv:math/0605527v5 [math.DS]

[25] M. L. Lapidus and C. Pomerance, The Riemann zeta-function and the one-dimensional Weyl-Berry conjecture for fractal drums, *Proc. London Math. Soc.* (3) 66 (1993), 4169.

[26] M. L. Lapidus and C. Pomerance, Counterexamples to the modified Weyl–Berry conjecture on fractal drums, *Math. Proc. Cambridge Philos. Soc.* 119 (1996), 167178.

[27] M.L. Lapidus and J. A. Rock, Toward zeta functions and complex dimensions of mutlifractals, Complex Variables and Elliptic Equations *(special issue dedicated to "Fractals")*, (6) 54 (2009), 545-549.

[28] K.S. Lau and S.M. Ngai. L^q spectrum of the Bernouilli convolution associated with the golden ration. *Studia Math.*, 131(3):225–251, 1998.

[29] J. Lévy Véhel and C. Tricot, On various multifractal spectra, *Fractal Geometry and Stochastics III, Progress in Probability*, C. Bandt, U. Mosco and M. Zähle (Eds), Birkhäuser Verlag, 2004, vol. 57, p. 23-42.

[30] J. Lévy Véhel and R. Vojak. Multifractal analysis of Choquet capacities. *Adv. in Appl. Math.*, 20(1):1–43, 1998.

[31] B.B. Mandelbrot, Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier, *J. fluid. Mech.*, 62 (1974), 331–358.

[32] J. R. Rock, Zeta functions, complex dimensions of fractal strings and multifractal analysis of mass distributions, PhD Thesis, UC Riverside, 2007.

[33] C. Tricot Curves and Fractal Dimension *Springer Verlag*, New York, (1995).