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Scaling and long range dependence in option pricing, IV: Pricing European options with transaction costs under the multifractional Black–Scholes model☆

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ABSTRACT

This paper deals with the problem of discrete time option pricing using the multifractional Black–Scholes model with transaction costs. Using a mean self-financing delta hedging argument in a discrete time setting, a European call option pricing formula is obtained. The minimal price of an option under transaction costs is obtained. In addition, we show that scaling and long range dependence have a significant impact on option pricing.
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1. Introduction

Over the last few years, the financial markets have been regarded as complex and nonlinear dynamic systems. A series of studies have found that many financial market time series display scaling laws and long range dependence. Therefore, it has been proposed that one should replace the Brownian motion in the classical Black–Scholes model [1] by a process with long range dependence. A simple modification is to introduce fractional Brownian motion (fBm) as the source of randomness. Thus one adds one parameter, H, to model the dependence structure in the stock prices (for references to these studies see Refs. [2–8]). However, empirical analysis of the ever-growing amount of data available from the financial markets revealed that the fBm does not allow one to take into account erosion phenomena in the sample paths of stock price movements, for the regularity of the fBm is the same along its whole path. This characteristic of the fBm is undesirable when we model those financial phenomena that need a time-varying Hurst exponent H_t (e.g., see Refs. [9–21] for details). To model the erosion phenomena in stock markets, a multifractional Brownian motion (mBm), with time-varying Hurst exponent $H_t = H(t)$ on the real line, has been proposed by Peltier and Levy-Vehel [22].

In this paper, on the basis of the points of view of behavioral finance [23,24] and econophysics [25] and empirical findings on the long range dependence in stock returns by Refs. [9–21], we will study the option pricing problem under transaction

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costs while the dynamics of the stock price S_t satisfies

$$S_t = S_0 e^{\mu t + \sigma B_{H_t}(t)}, \tag{1.1}$$

where $\mu, \sigma > 0$, and $S_0 > 0$ are constants; and $B_{H_t}(t)$ is a standard multifractional Brownian motion with time-varying Hurst exponent $H_t \in (1/3, 1)$. We will show that the time scaling and long range dependence in return series have an important impact on option pricing when proportional transaction costs are considered.

Leland [26] was the first who examined option replication in the presence of transaction costs in a discrete time setting. From the point of view of Leland [26], in a model where transaction costs are incurred every time the stock or the bond is traded, the arbitrage-free argument used by Black and Scholes [1] no longer applies. The problem is that due to the infinite variation of the geometric Brownian motion, perfect replication incurs an infinite amount of transaction costs. Hence, he suggested a delta hedge strategy incorporating transaction costs based on revision at a discrete number of times. Transaction costs lead to the failure of the no-arbitrage principle and the continuous time trade in general: instead of no arbitrage, the principle of hedge pricing – according to which the price of an option is defined as the minimum level of initial wealth needed to hedge the option – comes to the fore.

In Section 2, by using a delta hedging strategy, initiated by Leland [26], we will deduce an option pricing formula in the presence of transaction costs while the dynamics of the stock price S_t satisfies Eq. (1.1). In particular, we will analyze the impact of scaling and long range dependence on option pricing. In Section 3, a conclusion is given.

2. An option pricing model for a multifractional economy under transaction costs

2.1. The definition and property of multifractional Brownian motion

Let us be given, on a complete probability space (Ω, F, P), a multifractional Brownian motion $(W_{H_t}(t))$ with Hurst exponent $H_t \in (0, 1)$ as follows [22], where we assume that P is the real world probability measure.

Definition 2.1. Let $H_t = H(t) : [0, +\infty) \to (0, 1)$ be a Hölder function of exponent $\beta > 0$, i.e. for any $t_1, t_2 \in [0, +\infty)$ such that $|t_1 - t_2| < 1$, there exists a constant $c_0 > 0$ such that

$$|H(t_1) - H(t_2)| \le c_0 \, |t_1 - t_2|^{\beta} \, . \tag{2.1}$$

Then

$$W_{H_t}(t) = \frac{1}{\Gamma(H_t + 1/2)} \left[\int_{-\infty}^{0} ((t-\tau)^{H_t - 1/2} - (-\tau)^{H_t - 1/2}) \mathrm{d}W(\tau) + \int_0^t (t-\tau)^{H_t - 1/2} \mathrm{d}W(\tau) \right] \tag{2.2}$$

is called a multifractional Brownian motion (mBm), where $W(t)$ is a Brownian motion. Peltier and Levy [22] showed that a mBm has continuous sample paths with probability 1.

Proposition 2.1 ([22]). *With probability 1, W_{H_t} is a continuous function of t.*

Proposition 2.2 ([22]). *There exists a positive continuous function defined for $t \ge 0$; $t \to \sigma_t$ such that*

$$E\left[\frac{W_{H_{t+h}}(t+h) - W_{H_t}(t)}{h^{H_t}} \right] \to 0 \quad \text{as } h \to 0, \tag{2.3}$$

$$E\left[\frac{W_{H_{t+h}}(t+h) - W_{H_t}(t)}{h^{H_t}} \right]^2 \to \sigma_t^2 \quad \text{as } h \to 0, \tag{2.4}$$

and $\frac{W_{H_{t+h}}(t+h) - W_{H_t}(t)}{h^{H_t}} \to N(0, \sigma_t^2)$ *in distribution as* $h \to 0$.

Definition 2.2 (*Standard Multifractional Brownian Motion [22]*). Let $(W_{H_t}(t))_{t \ge 0}$ be a multifractional Brownian motion and let $t \to H_t$ be its Hölder functional parameter of exponent $\beta > 0$, such that for any $t \ge 0$, $0 < H_t < \min(1, \beta)$. Then there exists a unique continuous positive function $t \to \sigma_t$ such that the process $B_{H_t}(t) = \frac{W_{H_t}(t)}{\sigma_t}$ is continuous and satisfies

$$\mathrm{Var}\left(\frac{B_{H_{t+h}}(t+h) - B_{H_t}(t)}{h^{H_t}} \right) \to 1 \quad \text{as } h \to 0. \tag{2.5}$$

The process $(B_{H_t}(t))_{t \ge 0}$ is called a standard multifractional Brownian motion.

Remark 2.1. By an argument just like that of Peltier and Levy-Vehel in Ref. [22], we know that if a Hölder function H_t satisfies $H_t \in (\alpha, 1)$ then Propositions 2.1 and 2.2 and Eq. (2.5) still hold, where $0 < \alpha < 1$. In addition, from the dominated convergence theorem we know that if $H_t \in (\alpha, 1)$ and Eq. (2.5) holds then

$$\frac{B_{H_{t+h}}(t+h) - B_{H_t}(t)}{h^{H_t}} \to N(0, 1) \quad \text{in distribution as } h \to 0.$$

2.2. The pricing option for a multifractional economy under transaction costs

The groundwork of modeling the effects of transaction costs in option pricing was done by Leland [26]. He adopted the hedging strategy of rehedging at every time step, δt. That is, every δt, the portfolio is rebalanced, whether or not this is optimal in any sense. In the proportional transaction cost option pricing model, we follow the other usual assumptions of the Black–Scholes model but with the following exceptions:

(i) The price S_t of the underlying stock at time t satisfies a multifractional Black–Scholes model

$$S_t = S_0 \exp(\mu t + \sigma B_{H_t}(t)), \tag{2.6}$$

where $\mu, H_t > \frac{1}{3}, \sigma$ and $S_0 > 0$ are constants.

(ii) The portfolio is revised every δt, where δt is a finite and fixed, small time step.

(iii) Transaction costs are proportional to the value of the transaction in the underlying. Thus if υ shares are bought $(\upsilon > 0)$ or sold $(\upsilon < 0)$ at a price S_t, then the transaction cost is given by $\frac{k}{2}|\upsilon|S_t$ in either buying or selling, where k is a constant. The value of k will depend on the individual investor. In the multifractional Black–Scholes model where transaction costs are incurred at every time the stock or the bond is traded, the no-arbitrage argument used by Black and Scholes no longer applies. The problem is that due to the infinite variation of the geometric multifractional Brownian motion, perfect replication incurs an infinite amount of transaction costs.

(iv) The hedged portfolio has an expected return equal to that from an option. This is exactly the same valuation policy as earlier for discrete hedging without transaction costs.

(v) Traditional economics assumes that traders are rational and maximize their utility. However, if their behavior is assumed to be bounded rational, the traders' decisions can be explained both by their reaction to the past stock price, according to a standard speculative behavior, and by imitation of other traders' past decisions, according to common evidence in social psychology. It is well known that delta hedging strategies play a central role in the theory of option pricing. On the basis of the availability heuristic proposed by Tversky and Kahneman [27], traders are assumed to follow, anchor, and imitate the Black–Scholes delta hedging strategy to price an option. In this case, the delta hedging argument is a partial and imperfect hedging strategy, which does not eliminate all of the risk. However, as mentioned in the paper [25], in most models of stock fluctuations, except for very special cases, risk in option trading cannot be eliminated and strict arbitrage opportunities do not exist, whatever the price of the option. That the risk cannot be eliminated is furthermore the fundamental reason for the very existence of option markets.

Let $C = C(t, S_t)$ be the value of a European call on the above underlying stock at time t with expiration date T and exercise price X, and the boundary conditions

$$C(T, S_T) = (S_T - X)^+, \qquad C(t, 0) = 0, \qquad C(t, S_t) \to S_t \quad \text{as } S_t \to +\infty.$$

In addition, we assume that the riskless bond price dynamics satisfies

$$\mathrm{d}D(t) = rD(t)\mathrm{d}t \tag{2.7}$$

where r is a constant.

Consider a replicating portfolio with $X_1(t) = X_1(t, S_t)$ units of underlying asset and $X_2(t)$ units of the riskless bond. The value of the portfolio at current time t is

$$\Pi_t = X_1(t)S_t + X_2(t)D_t. \tag{2.8}$$

Next consider the changes in S_t and Π_t over the discrete time interval δt. After time interval δt, the change in the value of the underlying asset is

$$\delta S_t = S_t \left[\mathrm{e}^{\mu\delta t + \sigma \delta B_{H_t}(t)} - 1 \right]. \tag{2.9}$$

Since the current stock price S_t is a given constant and δt is small enough, from Remark 2.1 we get that

$$\begin{aligned} E[\delta S_t] &= S_t \left[\mathrm{e}^{\mu\delta t + \frac{\sigma^2}{2} E(\delta B_{H_t}(t))^2} - 1 \right] \\ &\approx S_t \left[\mu \delta t + \frac{\sigma^2}{2} (\delta t)^{2H_t} + O((\delta t + (\delta t)^{2H_t})^2) \right], \end{aligned} \tag{2.10}$$

$$\begin{aligned} E\left[(\delta S_t)^2\right] &= S_t^2 \left[\mathrm{e}^{2\mu\delta t + 2\sigma^2 E(\delta B_{H_t}(t))^2} - 2\mathrm{e}^{\mu\delta t + \frac{\sigma^2}{2} E(\delta B_{H_t}(t))^2} + 1 \right] \\ &\approx S_t^2 \left[\sigma^2 (\delta t)^{2H_t} + O((\delta t + (\delta t)^{2H_t})^2) \right], \end{aligned} \tag{2.11}$$

$$\begin{aligned} E\left[(\delta S_t)^4\right] &= S_t^4 \left[\sum_{j=0}^{4} \frac{4!}{j!\,(4-j)!} (-1)^{4-j} E\left(\mathrm{e}^{j\mu\delta t + j\sigma \delta B_{H_t}(t)} \right) \right] \\ &\approx S_t^4 \left[\sum_{j=0}^{4} \frac{4!}{j!\,(4-j)!} (-1)^{4-j} \mathrm{e}^{j\mu\delta t + \frac{j^2\sigma^2}{2} (\delta t)^{2H_t}} \right] \\ &= O((\delta t)^{4H_t}), \end{aligned} \tag{2.12}$$

and

$$E\left[(\delta S_t)^6\right] = S_t^6\left[\sum_{j=0}^{6}\frac{6!}{j!(6-j)!}(-1)^{6-j}E(e^{j\mu\delta t+j\sigma\delta B_{H_t}(t)})\right]$$
$$= S_t^6\left[\sum_{j=0}^{6}\frac{6!}{j!(6-j)!}(-1)^{6-j}e^{j\mu\delta t+\frac{j^2\sigma^2}{2}E(\delta B_{H_t}(t))^2}\right]$$
$$= O((\delta t)^{6H_t}). \tag{2.13}$$

The change in the value of the portfolio is

$$\delta\Pi_t = X_1(t)\delta S_t + X_2(t)\delta D_t - \frac{k}{2}\left|\delta X_1(t)\right| S_t, \tag{2.14}$$

where δD_t is the change in the riskless bond price, $\delta X_1(t)$ is the change in the number of units of asset held in the portfolio.

Since the time step and the asset price movement are both small, from Taylor's theorem we have

$$\delta D_t = rD_t\delta t + O((\delta t)^2), \tag{2.15}$$

$$\delta C(t,S_t) = \frac{\partial C(t,S_t)}{\partial t}\delta t + \frac{\partial C(t,S_t)}{\partial S_t}\delta S_t + \frac{1}{2}\frac{\partial^2 C(t,S_t)}{\partial S_t^2}(\delta S_t)^2 + \frac{1}{2}\frac{\partial^2 C(t,S_t)}{\partial t^2}(\delta t)^2 + \frac{\partial^2 C(t,S_t)}{\partial t\partial S_t}(\delta t)(\delta S_t) + G(\delta t),$$

where

$$G(\delta t) = \frac{1}{6}\left[\frac{\partial^3 C(\bar t,\bar S)}{\partial \bar t^3}(\delta t)^3 + 3\frac{\partial^3 C(\bar t,\bar S)}{\partial \bar t^2\partial \bar S}(\delta t)^2(\delta S_t) + 3\frac{\partial^3 C(\bar t,\bar S)}{\partial \bar t\partial \bar S^2}(\delta t)(\delta S_t)^2 + \frac{\partial^3 C(\bar t,\bar S)}{\partial \bar S^3}(\delta S_t)^3\right],$$

$\bar t = t + \theta_1\delta t, \bar S = S_t + \theta_1(\delta S_t), \theta_1 = \theta_1(t,\delta t,\omega), \omega \in \Omega$, and $0 < \theta_1 < 1$.

Note that since financial data are discrete in a real world we don't consider using "Ito's formula". The classical Ito formula only holds in a continuous time case.

Since the current stock price S_t is given, $\frac{\partial^3 C(t,S_t)}{\partial t^j\partial S_t^{3-j}}$ is also given ($j = 0, 1, 2, 3$). However, there exist long time correlations between $\frac{\partial^3 C(\bar t,\bar S)}{\partial \bar t^j\partial \bar S^{3-j}}$ and δS_t ($j = 0, 1, 2, 3$).

Assume that there exists a constant $M > 0$ such that

$$E\left(\frac{\partial^3 C(\bar t,\bar S)}{\partial \bar t^j\partial \bar S^{3-j}}\right)^2 < M^2 \quad (j = 0, 1, 2, 3).$$

Since the current stock price S_t is given, from Eqs. (2.11)–(2.13) we have

$$|E[G(\delta t)]| \le E|G(\delta t)| < \sum_{j=0}^{3}E\left|\frac{\partial^3 C(\bar t,\bar S)}{\partial \bar t^j\partial \bar S^{3-j}}(\delta t)^j(\delta S_t)^{3-j}\right|$$
$$\le \sum_{j=0}^{3}\left(E\left(\frac{\partial^3 C(\bar t,\bar S)}{\partial \bar t^j\partial \bar S^{3-j}}\right)^2\right)^{\frac{1}{2}}\left(E(\delta S_t)^{6-2j}\right)^{\frac{1}{2}}(\delta t)^j$$
$$< \sum_{j=0}^{3}M(\delta t)^j\left[E(\delta S_t)^{6-2j}\right]^{\frac{1}{2}}$$
$$= O\left((\delta t)^{3H_t}\right), \tag{2.16}$$

and

$$E[\delta C(t,S_t)] \approx \left(\frac{\partial C(t,S_t)}{\partial t} + \mu S_t\frac{\partial C(t,S_t)}{\partial S_t}\right)\delta t + \frac{\sigma^2}{2}S_t^2(\delta t)^{2H_t}\frac{\partial^2 C(t,S_t)}{\partial S_t^2} + \frac{\sigma^2}{2}(\delta t)^{2H_t}S_t\frac{\partial C(t,S_t)}{\partial S_t} + E[G(\delta t)] + O\left(\left(\delta t + (\delta t)^{2H_t}\right)^2\right). \tag{2.17}$$

Like for Eq. (2.17), from Taylor's theorem we know that there exists a constant $M_1 > 0$ such that

$$\delta X_1(t,S_t) = \frac{\partial X_1(t,S_t)}{\partial S_t}\delta S_t + \frac{\partial X_1(t,S_t)}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 X_1(t,S_t)}{\partial S_t^2}(\delta S_t)^2 + \frac{1}{2}\frac{\partial^2 X_1(t,S_t)}{\partial t^2}(\delta t)^2 + \frac{\partial^2 X_1(t,S_t)}{\partial t\partial S_t}(\delta t)(\delta S_t) + G_1(\delta t), \tag{2.18}$$

where

$$E\left|G_1\left(\delta t\right)\right| < \sum_{j=0}^{3} M_1\left(\delta t\right)^j \left[E\left(\delta S_t\right)^{6-2j}\right]^{\frac{1}{2}} \approx O\left(\left(\delta t\right)^{3H_t}\right). \tag{2.19}$$

Let

$$G_2(\delta t) = \frac{\partial X_1(t,S_t)}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 X_1(t,S_t)}{\partial S_t^2}\left(\delta S_t\right)^2 + \frac{1}{2}\frac{\partial^2 X_1(t,S_t)}{\partial t^2}\left(\delta t\right)^2 + \frac{\partial^2 X_1(t,S_t)}{\partial t \partial S_t}\left(\delta t\right)\left(\delta S_t\right). \tag{2.20}$$

From Eqs. (2.10) and (2.11), we know that

$$E\left|G_2\left(\delta t\right)\right| = O\left(\delta t\right). \tag{2.21}$$

Thus

$$\delta X_1(t,S_t) = \frac{\partial X_1(t,S_t)}{\partial S_t}\delta S_t + G_1(\delta t) + G_2(\delta t). \tag{2.22}$$

On the other hand from Taylor's theorem we have

$$\delta S_t = \sigma S_t \delta B_{H_t}(t) + \mu S_t \delta t + \frac{S_t}{2} e^{\theta\left(\mu\delta t + \sigma\delta B_{H_t}(t)\right)}\left(\mu\delta t + \sigma\delta B_{H_t}(t)\right)^2,$$

where $\theta = \theta\left(t, \delta t, \omega\right)$, $\omega \in \Omega$, and $0 < \theta < 1$.

Let $G_3(\delta t) = \mu S_t \delta t + \frac{S_t}{2} e^{\theta\left(\mu\delta t + \sigma\delta B_{H_t}(t)\right)}\left(\mu\delta t + \sigma\delta B_{H_t}(t)\right)^2$.

Then

$$\delta S_t = \sigma S_t \delta B_{H_t}(t) + G_3\left(\delta t\right), \tag{2.23}$$

and

$$E\left|G_3(\delta t)\right| \le |\mu| S_t \delta t + \frac{S_t}{2}\left(E\left(e^{2\left|\mu\delta t + \sigma\delta B_{H_t}(t)\right|}\right)\right)^{\frac{1}{2}}\left(E\left(\mu\delta t + \sigma\delta B_{H_t}(t)\right)^4\right)^{\frac{1}{2}} = O\left(\delta t\right). \tag{2.24}$$

Hence from Eqs. (2.18)–(2.24) we have

$$\delta X_1(t,S_t) = \sigma S_t \frac{\partial X_1(t,S_t)}{\partial S_t}\delta B_{H_t}\left(t\right) + \frac{\partial X_1(t,S_t)}{\partial S_t} G_3\left(\delta t\right) + G_1\left(\delta t\right) + G_2\left(\delta t\right). \tag{2.25}$$

So

$$E\left|\delta X_1(t,S_t)\right| \le \sigma S_t \left|\frac{\partial X_1(t,S_t)}{\partial S_t}\right| E\left|\delta B_{H_t}\left(t\right)\right| + E\left|\frac{\partial X_1(t,S_t)}{\partial S_t} G_3\left(\delta t\right)\right| + E\left|G_1\left(\delta t\right)\right| + E\left|G_2\left(\delta t\right)\right|, \tag{2.26}$$

and

$$E\left|\delta X_1(t,S_t)\right| \ge \sigma S_t \left|\frac{\partial X_1(t,S_t)}{\partial S_t}\right| E\left|\delta B_{H_t}\left(t\right)\right| - E\left|\frac{\partial X_1(t,S_t)}{\partial S_t} G_3\left(\delta t\right)\right| - E\left|G_1\left(\delta t\right)\right| - E\left|G_2\left(\delta t\right)\right|. \tag{2.27}$$

From Eqs. (2.19), (2.21), (2.24), (2.26) and (2.27), we obtain that

$$\lim_{\delta t \to 0} \frac{E\left|\delta X_1(t,S_t)\right|}{\left(\delta t\right)^{H_t}} = \sigma S_t \sqrt{\frac{2}{\pi}}\left|\frac{\partial X_1(t,S_t)}{\partial S_t}\right|.$$

Therefore

$$E\left|\delta X_1(t,S_t)\right| \approx \sigma S_t \sqrt{\frac{2}{\pi}}\left|\frac{\partial X_1(t,S_t)}{\partial S_t}\right|\left(\delta t\right)^{H_t}. \tag{2.28}$$

Let $C = C(t, S_t)$ be replicated by the portfolio Π_t. The value of the option must equal the value of the replicating portfolio Π_t to reduce (but not to avoid) arbitrage opportunities and be consistent with economic equilibrium.

Therefore

$$C(t,S_t) = X_1(t)S_t + X_2(t)D_t. \tag{2.29}$$

From the practical point of view, we assume that trading occurs at t and $t + \delta t$, but not in between. That means that the current stock price S_t and the number of shares given by the delta hedging strategy are held constant over the rebalancing interval $[t, t + \delta t)$.

Since the current stock price S_t is given, we know that $C(t, S_t)$, $\frac{\partial C(t,S_t)}{\partial t}$, $\frac{\partial C(t,S_t)}{\partial S_t}$ and $\frac{\partial C^2(t,S_t)}{\partial S_t^2}$ are constants, but different.

Thus, by the above assumptions (iv) and (v), $X_1(t) = \frac{\partial C(t,S_t)}{\partial S_t}$, Eqs. (2.10), (2.14), (2.16), (2.17), (2.19), (2.28) and (2.29), we set

$$\begin{aligned} E\left[\delta\Pi_t - \delta C_t\right] &= \left(rC(t,S_t) - \frac{\partial C(t,S_t)}{\partial t} - rS_t\frac{\partial C(t,S_t)}{\partial S_t} - \frac{\sigma^2}{2}S_t^2\left(\delta t\right)^{2H_t-1}\frac{\partial^2 C(t,S_t)}{\partial S_t^2}\right)\delta t \\ &\quad - \frac{\sigma k}{2}S_t^2\left|\frac{\partial^2 C(t,S_t)}{\partial S_t^2}\right|\sqrt{\frac{2}{\pi}}\left(\delta t\right)^{H_t} - E\left[G\left(\delta t\right)\right] - O\left(\left(\delta t + (\delta t)^{2H_t}\right)^2\right) \\ &= 0, \end{aligned} \tag{2.30}$$

which is a mean self-financing delta hedging strategy in a discrete time setting.

So from Eqs. (2.16) and (2.30) we have

$$rC = \frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S_t} + \frac{\sigma^2}{2}S_t^2\left(\delta t\right)^{2H_t-1}\frac{\partial^2 C}{\partial S_t^2} + \frac{\sigma k}{2}S_t^2\sqrt{\frac{2}{\pi}}\left(\delta t\right)^{H_t-1}\left|\frac{\partial^2 C}{\partial S_t^2}\right|. \tag{2.31}$$

Let

$$Le(H_t) = \frac{k}{\sigma\left(\delta t\right)^{1-H_t}}\sqrt{\frac{2}{\pi}}, \tag{2.32}$$

which is called multifractional Leland number.

From Eq. (2.31) we have

$$\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S_t} + \frac{\sigma^2}{2}S_t^2\left(\delta t\right)^{2H_t-1}\frac{\partial^2 C}{\partial S_t^2} + \frac{\sigma^2}{2}S_t^2\left|\frac{\partial^2 C}{\partial S_t^2}\right| Le(H_t) - rC = 0. \tag{2.33}$$

In particular, if $H_t = \frac{1}{2}$, from Eq. (2.33) we have

$$\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S_t} + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 C}{\partial S_t^2} + \frac{\sigma^2}{2}S_t^2\left|\frac{\partial^2 C}{\partial S_t^2}\right| Le\left(\frac{1}{2}\right) - rC = 0,$$

which is called Leland's equation for option pricing, and $Le\left(\frac{1}{2}\right)$ is called the Leland number.

Note that the additional term $\frac{\sigma^2}{2}S_t^2\left|\frac{\partial^2 C}{\partial S_t^2}\right| Le(H_t)$ is nonlinear, except when $\Gamma = \frac{\partial^2 C}{\partial S_t^2}$ does not change sign for all S_t. Since Γ represents the degree of mishedging of the portfolio, it is not surprising to observe that Γ is involved in the transaction cost term. We may rewrite Eq. (2.33) in a form which resembles the Black–Scholes equation:

$$\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S_t} + \frac{\tilde{\sigma}^2}{2}S_t^2\frac{\partial^2 C}{\partial S_t^2} - rC = 0, \tag{2.34}$$

where the modified volatility is given by

$$\tilde{\sigma} = \sigma\left[(\delta t)^{2H_t-1} + Le(H_t)sign(\Gamma)\right]^{\frac{1}{2}}. \tag{2.35}$$

If $\tilde{\sigma}^2$ becomes negative, Eq. (2.35) becomes mathematically ill-posed. This occurs when $\Gamma < 0$ and $Le(H_t) > (\delta t)^{2H_t-1}$. However, it is known that Γ is always positive for the simple European call and put options in the absence of transaction costs. If we postulate the same sign behavior for Γ in the presence of transaction costs, then

$$\tilde{\sigma}^2 = \sigma^2\left[(\delta t)^{2H_t-1} + Le(H_t)\right] > \sigma^2\left(\delta t\right)^{2H_t-1} > 0. \tag{2.36}$$

Now, Eq. (2.34) becomes linear under such assumptions so the Black–Scholes formulas become applicable except that the modified volatility $\tilde{\sigma}$ should be used as the volatility parameter.

Moreover, from Eq. (2.36) we obtain

$$C(t,S_t) = S_t N(d_1) - X\mathrm{e}^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln\left(S_t/X\right) + (r + \frac{\tilde{\sigma}_t^2}{2})(T-t)}{\tilde{\sigma}_t\sqrt{T-t}}, \qquad d_2 = d_1 - \tilde{\sigma}_t\sqrt{T-t},$$

$$\tilde{\sigma}_t^2 = \frac{\int_t^T \tilde{\sigma}^2 \mathrm{d}\tau}{T-t}, \qquad \tilde{\sigma}^2 = \sigma^2\left((\delta t)^{2H_\tau-1} + Le\left(H_\tau\right)\right), \tag{2.37}$$

and $N(\cdot)$ is the value of the cumulative normal density function.

Eq. (2.37) displays that the implicit volatility $\tilde{\sigma}_t$ varies with respect to t even if the volatility σ is a constant.

Furthermore, if $H_t = \frac{1}{2}$, and $k = 0$, from (2.34) we have

$$\frac{\partial C}{\partial t} + rS_t \frac{\partial C}{\partial S_t} + \frac{\sigma^2}{2} S_t^2 \frac{\partial^2 C}{\partial S_t^2} - rC = 0,$$

which is the Black–Scholes equation.

In particular, since $\frac{k}{\sigma} < \sqrt{\frac{\pi}{2}}$ often holds (for example: $\sigma = 0.3, k = 0.025$), from Eq. (2.36) we have

$$\frac{\tilde{\sigma}^2}{\sigma^2} = (\delta t)^{2H_t - 1} + \left(\frac{k}{\sigma}\sqrt{\frac{2}{\pi}}\right) \frac{1}{(\delta t)^{1-H_t}} \geq 2\left(\frac{2}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{k}{\sigma}}(\delta t)^{\frac{3}{2}H_t - 1}, \quad \text{where } H_t > \frac{1}{2}.$$

The minimal "volatility" $\tilde{\sigma}_{\min}$ is $\sigma\sqrt{2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}-\frac{1}{4H_t}}\left(\frac{k}{\sigma}\right)^{1-\frac{1}{2H_t}}$, as $\delta t = \left(\frac{2}{\pi}\right)^{\frac{1}{2H_t}}\left(\frac{k}{\sigma}\right)^{\frac{1}{H_t}}$, and $\min \tilde{\sigma}_t^2 = \frac{\int_t^T \tilde{\sigma}_{\min}^2 \mathrm{d}\tau}{T-t}$.

Thus the minimal price of an option under transaction costs is represented as $C_{\min}(t, S_t)$ with $\tilde{\sigma}_{\min}$ in Eq. (2.36), where $\tilde{\sigma}_{\min} = \sigma\sqrt{2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}-\frac{1}{4H_t}}\left(\frac{k}{\sigma}\right)^{1-\frac{1}{2H_t}}$ varies with respect to time t.

Furthermore, the option rehedging time interval for traders is $\delta t = \left(\frac{2}{\pi}\right)^{\frac{1}{2H_t}}\left(\frac{k}{\sigma}\right)^{\frac{1}{H_t}}$, which varies with respect to time t. The minimal price $C_{\min}(t, S_t)$ can be used as the actual price of an option.

On the other hand, we have

$$\tilde{\sigma}_{\min} = \sigma\sqrt{2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}-\frac{1}{4H_t}}\left(\frac{k}{\sigma}\right)^{1-\frac{1}{2H_t}} \to \sqrt{2}\sigma \quad \text{as } H_t \to \left(\frac{1}{2}\right)^+, \tag{2.38}$$

$$\text{and} \quad \delta t = \left(\frac{2}{\pi}\right)^{\frac{1}{2H_t}}\left(\frac{k}{\sigma}\right)^{\frac{1}{H_t}} \to \frac{2}{\pi}\left(\frac{k}{\sigma}\right)^2 \quad \text{as } H_t \to \left(\frac{1}{2}\right)^+. \tag{2.39}$$

In paper [28], Lux and Marchesi have shown that Hurst exponent $H_t = 0.51 \pm 0.004$ in some cases; therefore Eqs. (2.38) and (2.39) have a practical application in option pricing. For example: if $H_t \to 0.5^+$, $k = 1\%$ and $\sigma = 10\%$, then $\tilde{\sigma}_{\min} \to \frac{\sqrt{2}}{10}$, and $\delta t \to \frac{0.02}{\pi}$; and if $H_t \to 0.5^+$, $k = 0.1\%$ and $\sigma = 10\%$, then $\tilde{\sigma}_{\min} \to \frac{\sqrt{2}}{10}$, but $\delta t \to \frac{2}{\pi} \times 10^{-4}$.

In the following, we investigate the impact of scaling and long range dependence on option pricing. It is well known that Mantegna and Stanley [29,30] introduced the method of scaling invariance from complex science into economic systems for the first time. Since then, a lot of research on scaling laws in finance has begun. If $H_t = \frac{1}{2}$ and $k = 0$, from Eq. (2.36) we know that $\tilde{\sigma}^2 = \sigma^2 (\delta t)^{2H_t - 1} = \sigma^2$, which shows that fractal scaling δt has no impact on option pricing if a mean self-financing delta hedging strategy is applied in a discrete time setting. In particular, from Eq. (2.38) we know that $\tilde{\sigma}_{\min} \to \sqrt{2}\sigma$(as $H_t \to \frac{1}{2}$) is scaling invariant with respect to parameter k. On the other hand, if $H_t > \frac{1}{3}$ and $k \to 0$, from Eq. (2.36) we know that $\tilde{\sigma}^2 \to \sigma^2 (\delta t)^{2H_t - 1}$, which displays that the fractal scaling δt has a significant impact on option pricing. For $k = 0$ and $\delta t \to 0$, it is interesting that $\tilde{\sigma}^2 \to \sigma^2 (\delta t)^{2H_t - 1} \to 0$ if $H_t > \frac{1}{2}$ and $\tilde{\sigma}^2 \to \sigma^2 (\delta t)^{2H_t - 1} \to \infty$ if $H_t \in \left(\frac{1}{3}, \frac{1}{2}\right)$, which shows that $H_t = \frac{1}{2}$ is a critical point. Furthermore, for $k \neq 0$, from Eq. (2.36) we know that option pricing is scaling dependent in general.

Remark 2.2. Leland's delta hedging argument is an imperfect hedging strategy. In fact, even in the absence of transaction costs, discrete time rebalancing does not lead to perfect hedging, but nevertheless, imperfect hedging strategies have become a standard vehicle for evaluating derivatives in practice. Rehedging will reduce, but not eliminate, risk, but at a cost: we can no longer appeal to "no arbitrage". The hedge strategy of Leland is not optimal in the sense of perfect hedging, but from the point of view of Simon, individuals are governed by bounded rationality and they look for a satisfying result rather than the optimal one [31]. Therefore, our results are economically meaningful.

Remark 2.3. Ayache and Levy-Véhel [17] analyzed the log of the Nikkei 225 index during the period 01/01/1980 to 05/11/2000. They discovered that most values of the Hurst exponent H_t for that period are between 0.2 and 0.8. Muniandy and Lim [12] found that the daily lows of the US Dow Jones Industrial Average Stock Index exhibit local self-similarity with time-varying Hurst exponents that increase from roughly $H \approx 0.5$ to $H \approx 0.9$ during the period of 3 January 1995 until 31 May 2000. Tabak and Cajueiro [13] discovered that interest rates with maturities between 6 and 24 months present increasing Hurst exponents from 0.425 to 0.575 and that interest rates with maturities between 7 and 20 years present monotonically increasing Hurst exponents, from 0.525 to 0.60, for daily observations on Japanese interest rates over the period from July 10, 1992, to July 7, 2004. The analyses above are interesting and original, which show that estimating the Hurst regularity on the basis of a modeling with mBm yields novel insights into the data. Furthermore, Cajueiro and Tabak [18–21] have studied both developed and emerging stock markets and have presented a variety of results. They have made tests for time-varying long range dependence, which is known as a sign of multifractality, and suggested that there exists long range dependence in both mean and volatility of stock returns for a variety of countries. Their studies are important for two main reasons. Firstly, describing precisely the dynamics of the asset price is crucial for asset pricing. Secondly, evidence of long range dependence in asset returns and volatility has important implications for both portfolio and risk management.

Remark 2.4. From Ref. [32] we know that Eq. (2.37) is a formula for the long call price.

3. Conclusion

Without using an arbitrage argument, in this paper we obtain a European call option pricing formula with transaction costs for the multifractional Black–Scholes model with time-varying Hurst exponent $H_t \in \left(\frac{1}{3}, 1\right)$. It has been shown that the time scaling δt and Hurst exponent H_t play an important role in option pricing with transaction costs. In particular, for $H_t > \frac{1}{2}$ the minimal price of an option under transaction costs is obtained, which can be used as the actual price of an option. In addition, we also display that the implicit volatility $\tilde{\sigma}_t$ varies with respect to time t even if the volatility σ is a constant and that the option rehedging time interval $\delta t = \left(\frac{2}{\pi}\right)^{\frac{1}{2H_t}} \left(\frac{k}{\sigma}\right)^{\frac{1}{H_t}}$ varies with respect to time t.

References

[1] F. Black, M.S. Scholes, The pricing of option and corporate liabilities, J. Polit. Econ. 81 (1973) 637–659.
[2] B.B. Mandelbrot, The Fractal Geometry of Nature, W.H. Freeman, San Francisco, CA, 1982.
[3] J.-P. Bouchaud, D. Sornette, The Black–Scholes option pricing problem in mathematical finance: Generalization and extensions for a large class of stochastic process, J. Phys. France 4 (1994) 863–881.
[4] P. Cheridito, Arbitrage in fractional Brownian motion models, Finance Stoch. 7 (4) (2003) 533–553.
[5] A. Dasgupta, G. Kallianpur, Arbitrage opportunities for a class of Gladysher process, Appl. Math. Optim. 41 (3) (2000) 377–385.
[6] D.M. Salopek, Tolerance to arbitrage, Stoch. Process. Appl. 76 (2) (1998) 217–230.
[7] A.N. Shiryaev, On arbitrage and replication for fractal models. Technical report, MaPhySto, 1998.
[8] P. Guasoni, No arbitrage under transaction costs, with fractional Brownian motion and beyond, Math. Finance 16 (3) (2006) 569–582.
[9] Zeynel Abidin Ozdemir, Linkages between international stock markets: A multivariate long-memory approach, Physica A 388 (2009) 2461–2468.
[10] J.A. Ramirez, J. Alvarez, E. Rodriguez, G.F. Anaya, Time-varying Hurst exponent for US stock markets, Physica A 387 (2008) 6159–6569.
[11] Sunil Kumar, Nivedita Deo, Multifractal properties of the Indian financial market, Physica A 388 (2009) 1593–1602.
[12] S.V. Muniandy, S.C. Lim, Modeling of locally self-similar process using multifractional Brownian motion of Riemann–Liouville type, Phys. Rev. E 65 (2001) 0461104.
[13] B.M. Tabak, D.O. Cajueiro, The long-range dependence behavior of the term structure of interest rates in Japan, Physica A 350 (2005) 418–426.
[14] H. Erzgraber, F. Strozzi, J.-M. Zaldivar, H. Touchette, E. Gutierrez, D.K. Arrowsmith, Time series analysis and long range correlation of Nordic spot electricity market data, Physica A 387 (2008) 6567–6574.
[15] M.C. Mariani, I. Florescu, M.P. Beccar Varela, E. Ncheuguim, Long correlations and Levy models applied to the study of memory effects in high frequency data, Physica A 388 (2009) 1659–1664.
[16] S.V. Muniandy, S.C. Lim, R. Murugan, Inhomogeneous scaling behaviors in Malaysian foreign currency exchange rates, Physica A 301 (2001) 407–428.
[17] Antoine Ayache, Jacques Lévy Véhel, On the identification of the pointwise Hölder exponent of the generalized multifractional Brownian motion, Stoch. Process. Appl. 111 (2004) 119–156.
[18] Daniel O. Cajueiro, Benjamin M. Tabak, The Hurst exponent over time: Testing the assertion that emerging markets are becoming more efficient, Physica A 336 (2004) 521–537.
[19] Daniel O. Cajueiro, Benjamin M. Tabak, Evidence of long range dependence in Asian equity markets: The role of liquidity and market restrictions, Physica A 342 (2004) 656–664.
[20] Daniel O. Cajueiro, Benjamin M. Tabak, Testing for time-varying long-range dependence in real state equity returns, Chaos Solitons Fractals 38 (2008) 293–307.
[21] Daniel O. Cajueiro, Benjamin M. Tabak, Testing for long-range dependence in the Brazilian term structure of interest rates, Chaos Solitons Fractals 40 (2009) 1559–1573.
[22] R.F. Peltier, J. Levy-Vehel, INRIA Technical Report No. 2645, 1995.
[23] H. Shefrin, A behavioral approach to asset pricing, Elsevier Inc., 2005.
[24] A. Shleifer, Inefficient Markets: An Introduction to Behavioral Finance, Oxford University Press, 2000.
[25] M. Potters, J.-P. Bouchand, D. Sestovic, Hedged Monte Carlo: Low variance derivative pricing with objective probabilities, Physica A 289 (2001) 517–525.
[26] H.E. Leland, Option pricing and replication with transaction costs, J. Finance 40 (1985) 1283–1301.
[27] A. Tversky, D. Kahneman, Availability: A heuristic for judging frequency and probability, Cognitive Psychology. 5 (1973) 207–232.
[28] Thomas Lux, Michele Marchesi. Scaling and criticality in a stochastic multi-agent model of a financial market, Nature 397 (1999) 498–500.
[29] R.N. Mantegna, H.E. Stanley, Scaling behavior in the dynamics of an economic index, Nature 376 (1995) 46–49.
[30] R.N. Mantegna, H.E. Stanley, An Introduction to Econophysics, Cambridge University Press, Cambridge, 2000.
[31] H.A. Simon, A behavioral model of rational choice, Q. J. Economics 69 (1955) 91–118.
[32] Phelim P. Boyle, Ton Vorst. Option replication in-discrete time with transaction costs, J. Finance XLV (1) (1998) 271–293.